Apsidal Precession in the Two-Body Problem with Retarded Gravity – Application to the Solar System

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Abstract **— The present paper proves for the first time that retarded gravitational interaction leads in general to an apsidal precession in the two-body problem. For this, the retarded gravitational potential of a translating and rotating sphere is analytically derived to second order in v/c for a flat space-time metric. Integration over a complete Kepler orbit via a perturbation approach yields then the corresponding perihelion precession in analytical form as a function of the usual orbital elements and the radius and spin frequency of the central mass. Applied to the solar system, all planets show a retrograde precession due to the retardation effect, with Mercury and Mars having the highest values of -0.3"/cy and -0.002"/cy respectively (assuming the speed of gravity as the speed of light). With those values significantly above the error level of recent observational data for the solar system, the effect should therefore be highly relevant for the calculation of high precision ephemerides. Besides, it could offer a convenient method to constrain the speed of gravity and at the same time provide a further test of the theory of General Relativity.**

***Index Term s*— retarded potentials, two-body problem, speed of gravity, perihelion precession, solar system dynamics, ephemerides**

1. INTRODUCTION

Since the times of Newton, the calculation of planetary orbits has always been based on the assumption that all masses in the solar system interact instantaneously with each other. Whilst Newton's laws implicitly require an instantaneous interaction, several scientists subsequently proposed a finite speed of gravity, mainly in order to explain observed deviations from Newton's laws like the perihelion precession of Mercury. However, it was already argued in 1805 by Laplace (transl.1966) that a finite speed would cause planetary orbits to quickly become unstable due to the directional aberration, in contradiction to experience. Whilst this led Laplace to conclude that gravitational interaction occurs with a speed many orders of magnitude larger than the speed of light, if not infinitely fast, in modern times this apparent paradox is resolved within the framework of General Relativity. (see e.g. Carlip (2000)) .

However, it does in fact not require General Relativity to resolve the paradox as Laplace's argument (as illustrated by Fig.1) was inconsistent in the first place, even in a pre-relativistic setting

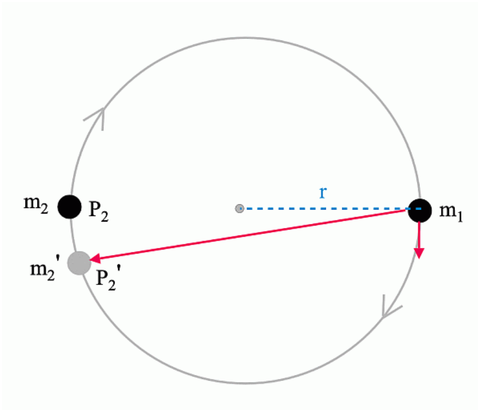

Fig1 Inconsistent model for retarded gravitational interaction using instantaneous center of mass as reference point

Here we have two equal masses m_1 and m_2 orbiting their common center of mass in a circular orbit. If the gravitational interaction is not instantaneous but happens with a finite speed, this would result in m_2 acting on m_1 not from its instantaneous position P_2 but from a position P_2' at an earlier point of the particle trajectory/orbit corresponding to the travel time of the interaction. If gravity would in this way be originating from a past 'ghost image' of m_2 rather than m_2 itself, this would, according to this argument, lead to a non-radial force component which would accelerate m_1 in its orbit, making the latter quickly unstable, contradictory to experience.

However, this argument is erroneous: first of all, the concept of a center of mass/gravity as used here implies already (instantaneous} Newtonian interaction, so it is inconsistent to use this in combination with a retarded interaction (which is a non-Newtonian concept in this respect). If the masses indeed interact with the opposite

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'ghost' mass, the center of mass should be defined with regard to this ghost mass as well, not with regard to the instantaneous mass (which becomes physically irrelevant here).

Secondly, and more to the point, it is in fact inconsistent to consider the situation in the center of mass frame when trying to figure out which forces are acting on m_1 due to the retarded interaction The retarded time and potential has to be evaluated from the view point of the target mass i.e. m_1 in this case. In electrodynamics, the Lienard-Wiechert potential is analogously derived by strictly considering the trajectory of the source particle in the reference frame of the test particle (this does not imply any dynamical consequences for the interaction between the two but merely represents the relative position of the source particle with regard to the test particle in a purely kinematical way). Although Fig.1 pretends to represent the positions of m_2 and m_2' as they appear to m_1, it does actually not do this, The correct graphic for describing the orbit of m_2 with regard to m_1 is shown in Fig. 2.

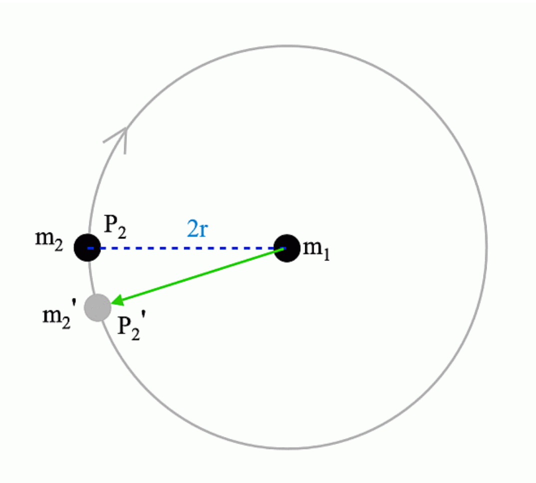

Fig2 Correct model for retarded two-body interaction with orbital motion referred to one of the masses

With regard to m_1 (the target mass), the radius of curvature of the orbit is actually $2r$ not r as suggested by Fig. 1, i.e. the vector from m_1 to any point on the orbit is always a radial vector. So the retarded position P_2' has in fact the same distance from m_1 as the instantaneous position P_2, contrary to what Fig.1 is suggesting. The latter (and thus the claim a finite speed of gravity would lead to unstable orbits) is thus only the result of inconsistently applying Newtonian concepts to a non-Newtonian problem. (non-Newtonian in the sense that the interaction is retarded, not that it has to be treated fully relativistically (this is termed 'propagation-delayed Newtonian gravity' by Carlip (2000))).

For circular orbits, like in the above illustrated case, m_1 will always see the same gravitational potential as the distance to any point on the relative orbit of m_2 is the same. This means that even for a retarded interaction, m_1 will in this case only see a radial force component (the same as for instantaneous interaction) and will thus not be dynamically affected by the speed of gravity. The only effect will be a certain phase offset in the orbit, otherwise there is no difference to the instantaneous case.

The calculation presented below will indeed confirm that dynamical changes only occur for eccentric orbits, because the retarded potential is only sensitive to changes in the radial distance of the two masses but not transverse motions. This is discussed for instance in detail in a recent paper by Smid (2019) showing the effect of a finite signal propagation speed on the apparent configuration of moving particle distributions. Equating the effect of retardation with directional aberration (as Carlip (2000) and others have done) misses therefore not only the point but leads to incorrect conclusions (whether or not there is actually an aberration). As is evident from Fig. 2, aberration as such does not change the distance and therefore not the gravitational potential (which is the quantity the present analysis is based on).

Possibly because of the incorrect (and irrelevant) argument regarding the aberration, the effect of retarded gravity on solar system dynamics has never been quantitatively determined, be it classically or within the framework of General Relativity. Many authors maintain that in the latter context retarded interaction would be implicitly contained in the relevant equations, but this is clearly incorrect, as is evident from the JPL publication by Moyer (1971): the non-Newtonian contribution to the 1-body problem (Eq.(20) there) is derived on the basis of the static Schwarzschild metric, in which the space and time intervals are understood as instantaneous quantities (as confirmed in personal communication with JPL) i.e. retarded gravity is not taken into account.. The present treatment shows however that the retardation effect should indeed lead to observable differences in the precession rates of some planets unless gravity propagates at least with more than ten times the speed of light (given present observational accuracy).

The theoretical basis for this work was laid already in the recent publication by Smid (2019) which solved analytically the retardation equation both for rectilinear and circular particle trajectories, this merely needed to be extended to be applicable to Kepler orbits. Additionally, the present approach also allows for a finite size as well as a rotation of the central mass. The resulting correction terms for the gravitational potential due to the retardation effect are then integrated along a complete Kepler orbit via a perturbation approach to yield the apsidal precession rate

An apsidal precession is indeed the only consequence to be expected from the finite propagation of gravity in the two-body problem as energy and angular momentum conservation can be assumed to hold (the system being a closed one) and thus the other orbital elements should not be affected.

1.1 RETARDED POTENTIAL OF POINT MASS

In order to calculate the retarded position of the source mass (the sun) with regard to the target mass (the planet), we assume the latter at the origin at the coordinate system and the instantaneous coordinates of the centre of the former as x, y. The source mass is assumed to be a spherical shell with en effective radius R rotating rigidly with angular velocity ω in the orbital (x, y) plane (R will be related to the actual radius of the mass at the final stage of the calculation depending on the model assumptions for the radial density structure),

A mass element of the shell has therefore the instantaneous coordinates

$$x_S = x + R\cdot\cos(\varphi)\cdot\sin(\vartheta) \tag{1}$$

$$y_S = y + R\cdot\sin(\varphi)\cdot\sin(\vartheta) \tag{2}$$

$$z_S = R\cdot\cos(\vartheta) \tag{3}$$

where φ is the instantaneous rotation angle.

For a rotation with constant angular frequency ω in the x, y plane, the corresponding retarded coordinates are then

$$x'_S = x' + R\cdot\cos(\varphi - \omega\cdot\Delta t)\cdot\sin(\vartheta) \tag{4}$$

$$y'_S = y' + R\cdot\sin(\varphi - \omega\cdot\Delta t)\cdot\sin(\vartheta) \tag{5}$$

$$z_S' = R\cdot\cos(\vartheta) \tag{6}$$

where

$$x' = x - v_x\cdot\Delta t - \frac{a_x}{2}\cdot\Delta t^2 \tag{7}$$

$$y' = y - v_y\cdot\Delta t - \frac{a_y}{2}\cdot\Delta t^2 \tag{8}$$

are the retarded coordinates of the centre of the rotating mass with regard to the planet, assuming the orbital velocity and acceleration vectors to be constant during the time interval Δt (the travel time for gravity between the corresponding mass element of the central mass and the planet) .

The signal travel time Δt is then obtained from the retardation equation

$$d'^2 = c^2\cdot\Delta t^2 \tag{9}$$

where

$$\mathbf{d'} = \mathbf{r_S}' = (x'_S\,,\, y'_S\,,\, z'_S) \quad . \tag{10}$$

Note that in (9) we use the symbol c for the speed of gravity as a free parameter i.e. without necessarily implying that this is identical to the speed of light. If c is indeed interpreted as the speed of light here, the retardation equation (9) is nothing but the equation for the Minkowski metric on the light cone so it should be fully appropriate to calculate the retardation effect in the weak field (almost-flat spacetime) limit of General Relativity (where distances can be calculated in the Euclidean metric with high accuracy).

If we use now a Taylor expansion for the trigonometric functions in (4) and (5) up to second order in $\omega\cdot\Delta t$, we can write for the square of the retarded distance (9)

$$d'^2 = d^2 \cdot \left(1 + \frac{f_1 \cdot \Delta t}{d^2} + \frac{f_2 \cdot \Delta t^2}{d^2} \right) \qquad (11)$$

where d is the instantaneous distance

$$d = \sqrt{x^2 + y^2 + \rho^2} \qquad (12)$$

with

$$\rho^2 = R^2 + 2R \cdot \sin(\vartheta) \cdot (x \cdot \cos(\varphi) + y \cdot \sin(\varphi)) \qquad (13)$$

(note that ρ^2 is merely a placeholder for the right hand side of (13) here (which is a combination of the coordinates for the mass element of the central mass and the coordinates of the test mass) and there is no particular meaning attached to the variable ρ as such)..

Furthermore in (11) we have

$$\begin{aligned} f_1 = &-2 \cdot (v_x \cdot x + v_y \cdot y) - \\ &-2R \cdot \sin(\vartheta) \cdot \cos(\varphi) \cdot (v_x + \omega \cdot y) - \\ &-2R \cdot \sin(\vartheta) \cdot \sin(\varphi) \cdot (v_y - \omega \cdot x) \end{aligned} \qquad (14)$$

$$\begin{aligned} f_2 = &\, v_x^2 + v_y^2 - a_x \cdot x - a_y \cdot y - \\ &- R \cdot \sin(\vartheta) \cdot \cos(\varphi) \cdot (a_x - 2\omega \cdot v_y + \omega^2 \cdot x) - \\ &- R \cdot \sin(\vartheta) \cdot \sin(\varphi) \cdot (a_y + 2\omega \cdot v_x + \omega^2 \cdot y) \end{aligned} \qquad (15)$$

Inserting (11) into (9) yields a quadratic equation for Δt which has the solution

$$\Delta t = \frac{f_1}{2 \cdot (c^2 - f_2)} \pm \frac{d}{c} \frac{\sqrt{1 + \frac{f_1^2}{d^2 \cdot (c^2 - f_2)}}}{\sqrt{1 - \frac{f_2}{c^2}}} \qquad (16)$$

Assuming the fractions under the square roots to be small against 1, we can use their Taylor expansion up to second order, which yields finally (choosing the positive sign in front of the square roots as we are interested in the retarded solution only and as Δt is defined as positive)

$$\Delta t = \frac{d}{c} \cdot \left(1 + \frac{f_1}{2 \cdot d \cdot c} + \frac{f_1^2}{2 \cdot d^2 \cdot c^2} + \frac{f_2}{2 \cdot c^2} \right) \qquad (17)$$

where in the final step the remaining expressions $c^2 - f_2$ in the denominator of the terms in the bracket where replaced by c^2 (which again means neglecting higher orders than v^2 / c^2 in the series approximation).

Eq.(17) gives directly the correction to the signal travel time due to the retardation effect in case of a moving source mass.

With Δt thus determined, we can then also derive the retarded potential from (11). First, taking the inverse of the square root we have

$$\frac{1}{d'} = \frac{1}{d} \cdot \frac{1}{\sqrt{1 + \frac{f_1 \cdot \Delta t}{d^2} + \frac{f_2 \cdot \Delta t^2}{d^2}}} \qquad (18)$$

and after a Taylor expansion of the denominator up to second order in Δt

$$\frac{1}{d'} = \frac{1}{d} \cdot \left(1 - \frac{f_1 \cdot \Delta t}{2 \cdot d^2} - \frac{f_2 \cdot \Delta t^2}{2 \cdot d^2} + \frac{3 \cdot f_1^2 \cdot \Delta t^2}{8 \cdot d^4} \right) \qquad (19)$$

Inserting (17) into (19) yields (keeping only terms up to c^2)

$$\frac{1}{d'} = \frac{1}{d} - \frac{f_2}{2 \cdot c^2 \cdot d} - \frac{f_1}{2 \cdot c \cdot d^2} + \frac{f_1^2}{8 \cdot c^2 \cdot d^3} \qquad (20)$$

If the gravitational interaction constant α (see below) is independent of the positions and velocities of the masses, equation (20) results therefore in a potential difference due to the retardation effect

$$\Delta u\left(\mathbf{r},\vartheta,\varphi\right)=-\alpha\cdot\left(\frac{1}{d'}-\frac{1}{d}\right)=$$
$$=\alpha\cdot\left(\frac{f_2}{2\cdot c^2\cdot d}+\frac{f_1}{2\cdot c\cdot d^2}-\frac{f_1^{\,2}}{8\cdot c^2\cdot d^3}\right)\qquad(21)$$

where

$$\alpha=G\cdot M\cdot m\qquad(22)$$

with G the gravitational constant, M the central (solar) mass and m the planet mass (we take M as the full central mass here, so later we merely average over ϑ and φ rather than integrate over differential mass elements). (note here that the term linear in f_1 (i..e. linear in v/c) averages to zero when integrated over a Kepler orbit as it is periodical (as will be shown in Sect.2.3), whilst for a circular orbit around a point mass all terms in (21) are indeed zero (as already qualitatively explained in the introduction)).

2.2 RETARDED POTENTIAL OF SPHERICAL SHELL

In order to obtain the potential of a spherical shell, we have to refer the instantaneous distance d (as defined in the previous section) to a mass element of the spherical (rotating) shell rather than the centre of the latter (to which we have to refer the orbital position later on), so we rewrite (12) as

$$d=r\cdot\sqrt{1+\frac{\rho^2}{r^2}}\qquad(23)$$

with

$$r=\sqrt{x^2+y^2}\qquad(24)$$

so we have

$$\frac{1}{d}=\frac{1}{r\cdot\sqrt{1+\frac{\rho^2}{r^2}}}\qquad(25)$$

Assuming now $|\rho^2|<<r^2$ we can expand the denominator up to order $|\rho^4|/r^4$ to yield

$$\frac{1}{d}=\frac{1}{r}\cdot(1-\frac{\rho^2}{2\cdot r^2}+\frac{3\cdot\rho^4}{8\cdot r^4})\qquad(26)$$

(note that the assumption $|\rho^2|<<r^2$ implies $R^2<<r^2$, so ρ^2 is actually dominated by the second (periodical) term in (13) which is proportional in order of magnitude to $R\cdot r$, i.e. the two small terms in (26) are actually merely of order R/r and R^2/r^2 respectively).

Similarly we get from (25) by means of series expansion of the denominator.

$$\frac{1}{d^2}=\frac{1}{r^2}\cdot(1-\frac{\rho^2}{r^2}+\frac{\rho^4}{r^4})\qquad(27)$$

$$\frac{1}{d^3}=\frac{1}{r^3}\cdot(1-\frac{3\cdot\rho^2}{2\cdot r^2}+\frac{15\cdot\rho^4}{8\cdot r^4})\qquad(28)$$

Inserting (26)-(28) into (21) we obtain

$$\Delta u(\mathbf{r},\vartheta,\varphi)=\alpha\cdot\left(\frac{f_1}{2\cdot c\cdot r^2}+\frac{f_1^{\,2}}{8\cdot c^2\cdot r^3}+\frac{f_2}{2\cdot c^2\cdot r}-\right.$$
$$-\frac{f_1\cdot\rho^2}{2\cdot c\cdot r^4}-\frac{3\cdot f_1^{\,2}\cdot\rho^2}{16\cdot c^2\cdot r^5}-\frac{f_2\cdot\rho^2}{4\cdot c^2\cdot r^3}+$$
$$\left.+\frac{f_1\cdot\rho^4}{2\cdot c\cdot r^6}+\frac{15\cdot f_1^{\,2}\cdot\rho^4}{64\cdot c^2\cdot r^7}+\frac{3\cdot f_2\cdot\rho^4}{16\cdot c^2\cdot r^5}\right)\qquad(29)$$

Averaging over ϑ,φ gives then the potential correction due to the retardation effect for a planet at location $\mathbf{r}$ due to a rotating spherical shell of radius R and mass M.

$$\Delta U(\mathbf{r},R)=\frac{1}{4\pi}\int_0^{\pi}\sin(\vartheta)\,d\vartheta\int_0^{2\pi}d\varphi\cdot\Delta u(\mathbf{r},\vartheta,\varphi)=$$

$$=\sum_{k=1}^{9}\Delta U_k \qquad (30)$$

where the individual terms resulting from the integration are given in Appendix A..1

2.3 PERIHELION PRECESSION DUE TO RETARDED POTENTIAL

In order to derive the apsidal precession resulting from the retarded potential correction (30) we use a perturbation approach outlined by Landau and Lifshitz (1976).. For this purpose, we briefly review here the corresponding equations for the 2-body problem in general and Keplerian orbits in particular.

For the reduced 2-body problem, the angular momentum and energy conservation laws can be respectively written as

$$\frac{d\phi}{dt}=\frac{L}{\mu\cdot r^2} \qquad (31)$$

$$E=\frac{\mu}{2}\cdot\left((\frac{dr}{dt})^2+r^2\cdot(\frac{d\phi}{dt})^2\right)+U(r')=$$
$$=\frac{\mu}{2}\cdot(\frac{dr}{dt})^2+\frac{L^2}{2\cdot\mu\cdot r^2}+U(r') \qquad (32)$$

where r,ϕ are the polar coordinates of the orbiting mass, L its angular momentum, $U(r')$ the potential energy, E the total energy and

$$\mu=\frac{m\cdot M}{m+M} \qquad (33)$$

the reduced mass for the two bodies with masses m and M .

.

Note that the potential energy is here, contrary to the usual formulation of the energy conservation law, a function of the retarded distance r' between the two masses rather than the instantaneous distance r .

For a circular orbit, we have obviously $r'=r$ and the retarded potential is thus always identical to the instantaneous one. In this case, the retardation has no effect at all on the orbital dynamics of the system as equations (31) and (32) are identical to the instantaneous case (see also the discussion in the introduction of this paper). The final result of the calculations will only confirm this conclusion.

Solving now (32) for dt and inserting into (31), we obtain

$$d\phi=L\cdot\frac{dr}{r^2\cdot\sqrt{2\mu\cdot(E-U(r'))-L^2/r^2}} \qquad (34)$$

If the motion is finite and we integrate from the minimum radius $r_{\min}$ to the maximum radius $r_{\max}$ and back we get thus from (34) for the corresponding angle covered

$$\phi_p:=2\cdot L\cdot\int_{r_{\min}}^{r_{\max}}\frac{dr}{r^2\cdot\sqrt{2\mu\cdot(E-U(r'))-L^2/r^2}} \qquad (35)$$

ϕ_p is thus the polar angle covered during a complete radial orbital period from perihelion to perihelion. For the motion in a 1/r potential (also for the harmonic oscillator potential ~r^2) this is exactly 2π and the orbit is closed, but for any other potential it will be different from this and the orbit will not be closed i.e. the line of apsides will be precessing. either forwards or backwards depending on the potential.

We first rewrite (35) as

$$\phi_p=-2\cdot\frac{\partial}{\partial L}\int_{r_{\min}}^{r_{\max}}dr\cdot\sqrt{2\mu\cdot(E-U(r'))-L^2/r^2} \qquad (36)$$

and now assume that $U(r')$ can be described by

$$U(r')=-\frac{\alpha}{r}+\Delta U(r') \qquad (37)$$

with α given by (22) here and $|\Delta U(r')|<<|\alpha/r|$.

Inserting (37) in (36) and expanding the integrand up to first order in $\Delta U(r')$ yields

$$\phi_p=2\pi+\Delta\phi \qquad (38)$$

where

$$\Delta\phi = 2\mu \cdot \frac{\partial}{\partial L} \int_{r_{\min}}^{r_{\max}} \frac{dr \cdot \Delta U(r')}{\sqrt{2\mu \cdot (E + \alpha / r\ ) - L^2 / r^2}} \qquad (39)$$

which, using (34), we can also simply write as

$$\Delta\phi = \mu \cdot \frac{\partial}{\partial L}\left(\frac{1}{L} \cdot \int_{0}^{2\pi} d\phi \cdot r(\phi)^2 \cdot \Delta U(r'(\phi), \phi) \right) \qquad (40)$$

(note that we have removed here the factor 2 introduced in (35) and instead integrated over 2π rather than π, as the perturbing potential $\Delta U(r'(\phi),\phi)$ is (unlike the unperturbed potential $-\alpha / r$ or the example potentials in Landau and Lifshitz (1976)) not identical in both halves of the orbit.).

In order to solve (40), we need to specify the functional dependence $r(\phi)$ for the 'unperturbed' orbit i.e. for $\Delta U(r'(\phi),\phi) = 0$ (which in this case means for instantaneous gravitational interaction). This is given by the well-known relationship for Kepler orbits

$$r(\phi) = \frac{p}{1 + e \cdot \cos(\phi)} \qquad (41)$$

where

$$p = \frac{L^2}{\mu \cdot \alpha} \qquad (42)$$

is a constant of motion (the 'semi latus rectum' distance}. ϕ in this context is also called the 'true anomaly', whilst e is the eccentricity

$$e = \sqrt{1 - \frac{b^2}{a^2}} \qquad (43)$$

where a and b are the semi-major and semi-minor axis of the ellipse respectively,

In order to perform the integration in (40) we also need to express $\Delta U(r')$ and thus the position, velocity and acceleration vectors for the expansion terms in Appendix A.1. as a function of ϕ. Again, from the standard equation for a Kepler orbit we have

$$x(\phi) = \frac{p \cdot \cos(\phi)}{1 + e \cdot \cos(\phi)} \qquad (44)$$

$$y(\phi) = \frac{p \cdot \sin(\phi)}{1 + e \cdot \cos(\phi)} \qquad (45)$$

The velocity and acceleration vectors can here not simply be obtained by differentiating x and y but have to be determined consistent with (7) and (8) where they were assumed as constant within the time interval Δt.For this reason we have to consider the retarded coordinates

$$x(\phi - \Delta\phi) = \frac{p \cdot \cos(\phi - \Delta\phi)}{1 + e \cdot \cos(\phi - \Delta\phi)} \qquad (46)$$

$$y(\phi - \Delta\phi) = \frac{p \cdot \sin(\phi - \Delta\phi)}{1 + e \cdot \cos(\phi - \Delta\phi)} \qquad (47)$$

where $\Delta\phi$ is the change in the true anomaly during time Δt

Using the addition theorem for the trigonometric functions, and expanding in terms of $\Delta\phi$ up to second order yields

$$\Delta x := x(\phi) - x(\phi - \Delta\phi) =$$

$$= -\frac{p \cdot \sin(\phi)}{(1 + e \cdot \cos(\phi))^2} \cdot \Delta\phi +$$

$$+ \frac{p \cdot \left(\cos(\phi) + e \cdot (1 + \sin^2(\phi)\right)}{2 \cdot \left(1 + e \cdot \cos(\phi)\right)^3} \cdot \Delta\phi^2 \qquad (48)$$

$$\Delta y := y(\phi) - y(\phi - \Delta\phi) =$$

$$= \frac{p \cdot (\cos(\phi) + e)}{(1 + e \cdot \cos(\phi))^2} \cdot \Delta\phi +$$

.

$$+ \frac{p \cdot \sin(\phi) \cdot \left(1 - e \cdot \cos(\phi) - 2 \cdot e^2\right)}{2 \cdot \left(1 + e \cdot \cos(\phi)\right)^3} \cdot \Delta\phi^2 \qquad (49)$$

By means of Kepler's second law (which is equivalent to the angular momentum conservation law (31))

$$\frac{\Delta\phi}{\Delta t} = \frac{a \cdot b}{r^2} \cdot \frac{2\pi}{T} \quad (50)$$

(where T the orbital period) and using furthermore (41) as well as the relationship

$$p = \frac{b^2}{a} \quad (51)$$

we can rewrite (48) and (49) as

$$\Delta x = -\frac{2\pi \cdot b \cdot \sin(\phi)}{(1-e^2) \cdot T} \cdot \Delta t +$$

$$+\frac{2\pi^2 p \cdot (1+e \cdot \cos(\phi)) \cdot \left(\cos(\phi) + e \cdot (1+\sin^2(\phi)\right)}{(1-e^2)^3 \cdot T^2} \cdot \Delta t^2 \quad (52)$$

$$\Delta y = \frac{2\pi \cdot b \cdot (\cos(\phi) + e)}{(1-e^2) \cdot T} \cdot \Delta t +$$

$$+\frac{2\pi^2 p \cdot (1+e \cdot \cos(\phi)) \cdot \sin(\phi) \cdot \left(1 - e \cdot \cos(\phi) - e^2\right)}{(1-e^2)^3 \cdot T^2} \cdot \Delta t^2 \quad (53)$$

By comparing (52) and (53) with (7) and (8) in the form

$$\Delta x = x - x' = v_x \cdot \Delta t + \frac{a_x}{2} \cdot \Delta t^2 \quad (54)$$

$$\Delta y = y - y' = v_y \cdot \Delta t + \frac{a_y}{2} \cdot \Delta t^2 \quad (55)$$

we see that

$$v_x = -\frac{2\pi \cdot b \cdot \sin(\phi)}{(1-e^2) \cdot T} \quad (56)$$

$$v_y = \frac{2\pi \cdot b \cdot (\cos(\phi) + e)}{(1-e^2) \cdot T} \quad (57)$$

$$a_x = \frac{4\pi^2 p \cdot (1+e \cdot \cos(\phi)) \cdot \left(\cos(\phi) + e \cdot (1+\sin^2(\phi)\right)}{(1-e^2)^3 \cdot T^2} \quad (58)$$

$$a_y = \frac{4\pi^2 p \cdot (1+e \cdot \cos(\phi)) \cdot \sin(\phi) \cdot \left(1 - e \cdot \cos(\phi) - e^2\right)}{(1-e^2)^3 \cdot T^2} \quad (59)$$

Note that the sign of the acceleration vector derived here is actually opposite to that of the actual (instantaneous) orbital acceleration vector. This is merely a consequence of the fact that we calculate the retarded from the instantaneous position i.e. going against the orbital motion. As a result, the linear velocity term in (54), (55) 'overshoots' the retarded position due to the curvature of the orbit and has to be 'pegged back' by the acceleration term in this second order approximation.

With this, we have now determined all variables required for evaluating the potential terms related to the retardation terms (as given in Appendix A1) and can calculate the resulting perihelion precession of the planetary orbits by means of (40).. We merely have to additionally use Kepler's third law

$$T = 2\pi \cdot \sqrt{\frac{\mu \cdot a^3}{\alpha}} \quad (60)$$

(with α and μ given by (22) and (33) respectively) and furthermore replace all occurrences of the orbital length constants a, b and p through the angular momentum L by means of (42), (43) and (51) in order to be able to differentiate with regard to L in (40). Doing this (using Mathematica) by adding up all the terms shown in Appendix A.1, then re-substituting the orbital parameters again for L by using (42) yields the following expression for the precession angle (in radians) per orbit

$$\Delta\phi = \Delta\phi_o + \Delta\phi_R + \Delta\phi_\omega \quad (61)$$

where

$$\Delta\phi_o = -\frac{2\pi \cdot G \cdot (M+m) \cdot \left(1 - \sqrt{1-e^2}\right)}{c^2 \cdot p} \quad (62)$$

is the precession due to the orbital eccentricity (this is the only term independent of the radius R of the central mass M i.e. the only non-zero term in the case of point masses), Note that $\Delta\phi_o$ is always negative i.e. the precession due to the retarded orbital position is always retrograde. This circumstance can be qualitatively explained by the fact that the retardation effect causes the probability density in the receding section of an orbit (as seen from any reference point) to be higher than in the approaching section (Smid (2019)). In the latter reference, the shown examples were for a reference point external to a circular rotating ring, but the result is qualitatively the same for an internal reference point. Fig. 3 shows the result for the reference point at $x = 0.5$ and with an anti-clockwise rotational velocity $v_r = 0.6c$

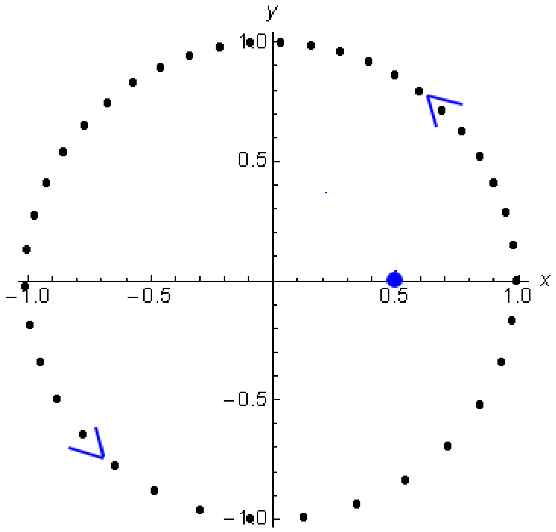

Fig.3 Schematic representation of retarded probability density for a planet in an anti-clockwise orbit with non-zero eccentricity (perihelion at x=1)

One can interpret this diagram schematically as representing the probability density for a planet to be found in a given part of an eccentric orbit (with the perihelion at x=1 and the aphelion at x=-1 and the focus (the sun) at x=0.5).. It shows the density to be higher in the top half (outgoing part) than in the bottom half (incoming part) of the orbit. This could obviously not be a stable orbit as the time spent in the two symmetric halves of the orbit would be different, Therefore the line of apsides (in this case the x-axis) has to rotate backwards (against the orbital motion) in order to re-establish symmetry between the two halves. This explains. the retrograde precession associated with the retardation effect due to the orbital velocity in an eccentric orbit as given by (62)

The term

$$\Delta\phi_R = \frac{5\pi \cdot G \cdot (M+m)}{2 \cdot c^2 \cdot p} \cdot \left[\frac{23 \cdot e^2 \cdot R^2}{3 \cdot p^2} + \right.$$

$$+ \frac{3 \cdot \left(9 + \frac{219}{4} \cdot e^2 + \frac{387}{16} \cdot e^4\right) \cdot R^4}{5 \cdot p^4} +$$

$$\left. + \frac{13 \cdot \left(1 + 11 \cdot e^2 + \frac{115}{8} \cdot e^4 + \frac{5}{2} \cdot e^6\right) \cdot R^6}{4 \cdot p^6} \right] \quad (63)$$

is the precession rate due to the finite radius R of the central mass (again, the mass represents here merely the orbital velocity). In contrast to the monopole term (62), the precession is here always positive but decreases much faster with distance. Referring again to Fig. 3, the positive sign of this term can be explained by the extended size of the central mass reducing the asymmetry between the two halves of the orbit, thus reducing the amount of retrograde precession due to the retardation effect associated with the monopole term (62). Note that this term is non-zero even for circular orbits ($e = 0$) due to some terms of order R^4 / p^4 and higher.

The third term

$$\Delta\phi_\omega = \frac{9\pi \cdot R^2 \cdot \omega^2}{2 \cdot c^2} \cdot \cdot \left[\frac{R^2}{p^2} + \frac{35 \cdot R^4}{36 \cdot p^4} \left(1 + \frac{3}{2} \cdot e^2\right) \right] -$$

$$- \frac{9\pi \cdot \sqrt{G \cdot (M+m)} \cdot \omega \cdot \sqrt{p}}{2 \cdot c^2} \cdot \left[\frac{2R^4}{p^4} (2 + 3e^2) + \right.$$

$$\left. + \frac{25 \cdot R^6}{9 \cdot p^6} \left(1 + 5 \cdot e^2 + \frac{15}{8} e^4\right) \cdot \right] \quad (64)$$

is the precession rate due to the rotation of the mass M . It consists of two terms, one being always positive and depending only on the rotational velocity $R \cdot \omega$ and the distance, the second being a cross term between the rotational and orbital velocity, which is in contrast always negative but decreases much faster with distance. This means that for large enough distances $\Delta\phi_\omega$ will change sign from a negative to a positive one. The numerical results will show that for the solar system this happens when going from Mars to Jupiter (assuming the speed of gravity is identical with the speed of light).

It may not be immediately obvious from the equations, but the overall precession due the retardation effect (61) is exactly zero for a circular orbit ($e = 0$) if the rotational angular frequency is equal to the orbital frequency i.e. if $\omega = 2\pi / T$, as in this case the terms in (63) and (64) not depending on e cancel exactly. This is of course expected as then each mass element of the rotating mass has always the same distance to the planet and the retardation will have no effect. In general though, it is clear that the retarded interaction leads to a coupling between the orbital and rotational parameters of the masses which should for instance lead to an additional tidal locking effect.. Consideration of the gravitational retardation effect may therefore well be necessary in order to fully understand the development of gravitational systems in this respect.

It should again be pointed out again that these equations have been obtained on the basis of second order series approximations in (effectively) the orbital or rotational velocity ratios v / c and the size/distance ratio R / p . As only even powers in the series development contribute to the results (the odd powers, representing periodical variations, average to zero during the orbit integration in (40)), the equations have in fact a relative accuracy of the order of v^2 / c^2 and $R /^2 p^2$ .and can therefore be considered exact in the present case as the accuracy of some of the physical constants going into the computation is a couple of orders of magnitude worse.

For realistic theoretical results that can be compared to observations, we have now only to additionally adjust the terms depending on the radius R of the central mass for the fact that the latter is not a spherical shell (which effectively assumed a density structure in form of a radial delta-function $\delta(R'-R)$) but a sphere with a continuous radial density structure. We can represent the latter here through a radial density function $\nu(\rho)$, where

$$\rho := \frac{R}{R_0} \qquad (65)$$

is the fractional radial distance from the centre of the sun with regard to its outer radius R_0 .($0 \le \rho \le 1$)..

As the surface of a shell with radius ρ is proportional to ρ^2 , we can define a corresponding shell mass function

$$\mu(\rho) := \rho^2 \cdot \nu(\rho) \qquad (66)$$

(not to be confused with the reduced mass defined in (33)), for which the normalization condition holds

$$\int_0^1 d\rho \cdot \mu(\rho) = 1 \qquad (67)$$

i.e. the total mass integrated over all shells up to the outer radius R_0 .must be M (the mass assumed for the shell of radius R .in our delta-function approximation for the radial density distribution in our model)..

With this, we are able to re-interpret the variable R .in (63) and (64) as the effective radius in our shell model. As R .however appears only in the powers 2, 4 and 6 here, we have to evaluate the corresponding moments of the radial mass function i.e. the integrals

$$R^k = R_0{}^k \cdot \int_0^1 d\rho \cdot \rho^k \mu(\rho) \qquad (68)$$

where k=2,4,6.

3. NUMERICAL RESULTS AND DISCUSSION

For evaluating the precession rates as given by (62)-(64), the masses (or rather the GM values) of the sun and planets (Mercury to Neptune) were taken from the DE430 ephemerides, Table 8 (Folkner et al. (2014)) and the Keplerian orbital elements (the semi-major axis and the eccentricity) from Table 8.10.2 in Standish and Williams (2012)).. In the latter case only the significant digits not affected by the indicated secular variations were taken, which amounted to about 3-5. The speed of gravity c was assumed here as the speed of light, also taken from the DE430 ephemerides.(Table 4), as was the solar radius R_0 (Table 9).

The radial density function $\nu(\rho)$ of the sun (see equation (65)) was taken here from the BS2005 table for the standard model of the sun (Bahcall et al. (2005)), and the corresponding mass function $\mu(\rho)$ and the values of R^k calculated from this via (66)-(68) by summing up all the correspondingly normalized and weighted table elements. This resulted in

$$R^2 = R_0{}^2 \cdot 0.1089$$

$$R^4 = R_0{}^4 \cdot 0.02797 \qquad (69)$$

$$R^6 = R_0{}^6 \cdot 0.01156$$

which was inserted in (63) and (64) (in the latter equation for the factored-out powers of R).

As the theoretical model assumes here a rigid rotation of the central mass, the rotational period of the sun was taken as 27 days (an average value considering the latitudinal variation of the solar rotation).

From the precession angles (62)-(64) relating to the change of the perihelion position over the course of one complete orbit, the usually quoted figure relating to a period of a century is obtained by

$$\Delta\Phi := \frac{\Delta\phi}{2\pi} \cdot 360 \cdot 3600 \frac{31{,}557{,}600}{\mathrm{T}} \cdot 100 \text{ [arcsec/cy]} \qquad (70)$$

where T is the orbital period in seconds as given by (60).

The results are shown in the Table 1.

	$\Delta\Phi_o$	$\Delta\Phi_R$	$\Delta\Phi_\omega$	$\sum$
Mercury	-3.061(-1)	+9,962(-6)	-4.095(-8)	**-3.060(-1)**
Venus	-6.453(-5)	+6.502(-9)	-1.098(-9)	**-6.452(-5)**
Earth/Moon	-1.784(-4)	+8.179(-9)	-1.503(-10)	**-1.784(-4)**
Mars	-1.964(-3)	+3.882(-8)	-5.655(-13)	**-1.964(-3)**
Jupiter	-2.397(-5)	+4.017(-11)	+4.608(-13)	**-2.397(-5)**
Saturn	-6.423(-6)	+3.207(-12)	+6.149(-14)	**-6.423(-6)**
Uranus	-8.870(-7)	+1.094(-13)	+5.557-15)	**-8.870(-7)**
Neptune	-9.329(-9)	+4.665(-16)	+1.161(-15)	**-9.329(-9)**

Note : (x)= *10^x , all figures in arcsec/cy

Table 1 Precession rates due to retarded gravitational sun/planet interaction with the speed of light c, for the individual terms as given by equations (62),(63),(64) + (70) and total.(Σ).

As is evident from (62), $\Delta\Phi_o$ (the only term not depending on size or rotation of the sun) is always negative i.e. the precession is retrograde, with Mercury having by far the highest value of about -0.3 arcseconds/cy, Mars is the second highest (due to the high eccentricity of its orbit), and the other planets 1-8 orders of magnitude smaller than this, decreasing inversely proportional with distance and varying additionally due to the dependence on the eccentricity (with this term only being non-zero for $e > 0$).

The multipole contribution $\Delta\Phi_R$ due to the finite size of the sun is in contrast always positive but several orders of magnitude smaller because of the small factors R^n / p^n.applied to the contributing terms in (63) (note that the term with R^2 / p^2 disappears for $e = 0$, but the higher order terms contribute also in case of a circular orbit).

The term $\Delta\Phi_\omega$ due to the rotation of the sun consists (as already mentioned below (64)) of a negative term coupled to the orbital velocity and a positive term solely due to the sun's rotational velocity. The latter term decreases less quickly with distance and dominates for larger distances. In this case it means that $\Delta\Phi_\omega$ is negative for the inner planets (up to Mars) and positive for the outer planets. However, the absolute value of this term is so small (less than a micro-arcsecond even for Mercury) that it is hardly observable, even if the speed of gravity would be much less than c.

The total precession due to the retardation effect of solar gravity is thus almost solely given by the orbital point mass term $\Delta\Phi_o$ and results in a retrograde rotation of the line of apsides for all planets. For Mercury and Mars (the planets with the highest eccentricities) the effect is large enough that it should be considered in corresponding orbital modeling. Yet from the recent publication of Park et al. (2017) (which is based on the same algorithm used to produce the DE430 ephemerides) it is evident that a retardation effect of this magnitude is not observed. The precession rate of Mercury's orbit derived in their study is solely composed of Newtonian contributions plus the static Schwarzschild and Lense-Thirring effects from General Relativity. (the planetary contributions differ from the Newtonian results in Stewart (2004) only by about 10% of the retardation effect derived here (of which 9% are due to the fact that in the latter publication the masses of Jupiter's and Saturn's moon have apparently been neglected), so the Sun-Mercury retardation effect could not have inadvertently been included in the planetary contributions in Park et al. (2017)). As the space-time curvature due to the mass of the sun would only introduce corrections to the Euclidean retardation values of the order of v^2/c^2, the precession due to the retardation effect resulting from the full PPN- equation of motion should essentially be the same as that derived in this work. The incompatibility of this term with the analysis of Park et al. (2017) (assuming c as the speed of light) raises therefore the question whether retarded gravity has in fact been correctly taken into account in their algorithm, or whether indeed the speed of gravity could be much higher than the speed of light (at least as far as the solar system is concerned, where as yet the speed of gravity could not be conclusively confirmed by observations).

It would be interesting to see a similar analysis to that in Park et al. (2017) for the case of Mars (and the other planets), but these do not appear to have been undertaken or published yet, although at least for Mars high accuracy data are available as well at JPL/NASA due to VLBI and spacecraft measurements (Folkner et al. (2014)).

The equations obtained here show that the precession effect due to retarded gravity increases strongly with increasing orbital velocity as well as eccentricity. The effect should therefore be even much stronger for instance for close binary stars in an eccentric orbit. For the well known Hulse-Taylor binary pulsar (PSR 1913+16) one obtains indeed, after inserting the relevant mass and orbital data (Taylor and Weisberg (1989)) into the above equations, a (retrograde) precession of -0.3 deg/y (assuming the speed of gravity as c) , which amounts to almost 10% of the measured total precession of +4.22 deg/y. (the effect of the rotation is even more negligible here than for the solar system due to the small size of the pulsar). Considering that the measurements are claimed to have a relative accuracy of at least 10^{-5}, this would therefore be highly relevant and could provide a further constraint for the speed of gravity.

Considering the results of this paper, it is clear that a correct interpretation of the fine details of observed secular precession rates is in general not possible without having the question regarding the speed of gravity conclusively answered one way or another, and the corresponding retardation effect properly being taken into account in the solar system modeling. In this sense, the conclusion of Iorio (2017) for instance that Verlinde's theory of emergent gravity could be tested by observations of the secular precession rates of Mars and Mercury must be seen as premature, as the retardation effect derived in this paper could potentially mask any small non-standard effect if it is not explicitly included in the modeling.

4. CONCLUSIONS AND OUTLOOK

The present work has derived analytical expressions for the retarded gravitational potential of a translating and rotating spherical mass on the basis of a flat spacetime metric approximation. Integrating these over a complete Kepler orbit via a perturbation approach has proved conclusively that a finite speed of gravity leads to a precession of the apsides in the gravitational 2-body problem. Applied to the planetary orbits in the solar system, the resulting precession turns out to be negative (i.e. retrograde) for all planets, being predominantly determined by the square of the ratio (orbital velocity/ speed of gravity) and eccentricity. If the speed of gravity is taken as the speed of light (as General Relativity requires), .the numerical value for the precession rate exceeds the claimed observational accuracy by far for some planets and should therefore be readily evident in corresponding data. However, a recent publication of Park et al. (2017) regarding the precession of Mercury's orbit shows no trace of the retardation effect, modeling the precession through Newtonian plus static GR terms only. This would indicate a mis-modeling of the solar system dynamics well above their claimed error level. Only if the speed of gravity is at least about 10 times the speed of light, would the retardation effect become insignificant for Mercury's secular precession rate, which however (like any speed different from c) would be problematic to reconcile with the theory of General Relativity. (despite the fact that the retarded potential only depends on radial motion, not on directional aberration as implied by Carlip (2000) and others)..
Whatever the reason for this apparent inconsistency is, accurate observations of precession rates of planetary orbits should be an as yet unrecognized and convenient way to confirm the speed of gravity from solar system dynamics. Using accurate precession measurements of binary star orbits (as it is possible for instance for systems containing a pulsar) the constraint for the value of the speed of gravity could be pushed even further.

5. DATA AVAILABILITY

No new data were generated or analysed in support of this research.

6. CONFLICT OF INTEREST

The authors declare that they have no conflict of interest

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APPENDIX A.1

EXPANSION TERMS FOR THE RETARDED POTENTIAL CORRECTION

Inserting (29) into (30) gives the following expansion terms for the gravitational potential correction (exact up to order v^2/c^2 in the orbital and rotational velocities and to order R^2/r^2 in the size of the central mass). For identification, the first 3 terms are due to the terms linear in f_1 in (29) (the first term in each line; these are periodical and vanish in the orbit integration). The terms 4-9 are the remaining terms in sequence as in (29). Of these, only ΔU_4 and ΔU_5 are non-zero for a central point mass. All the other terns are due the finite radius of the central mass and/or its rotation..

$$\Delta U_1 = -\alpha \cdot \frac{v_x x + v_y y}{c \cdot r^2} \quad \textbf{(A.1.1)}$$

$$\Delta U_2 = \alpha \cdot \frac{5 \cdot R^2 \cdot (v_x x + v_y y)}{3 \cdot c \cdot r^4} \quad \textbf{(A.1.2)}$$

$$\Delta U_3 = -\alpha \cdot \frac{R^2 \cdot (v_x x + v_y y) \cdot (7 \cdot R^2 + 4 \cdot r^2)}{3 \cdot c \cdot r^6} \quad \textbf{(A.1.3)}$$

$$\Delta U_4 = -\alpha \cdot \frac{3 \cdot (v_x x + v_y y)^2 + R^2 \cdot \left((v_x + \omega \cdot y)^2 + (v_y - \omega \cdot x)^2 \right)}{6 \cdot c^2 \cdot r^3} \quad \textbf{(A.1.4)}$$

$$\Delta U_5 = \alpha \cdot \frac{v_x^2 + v_y^2 - a_x x - a_y y}{2 \cdot c^2 \cdot r} \quad \textbf{(A.1.5)}$$

$$\Delta U_6 = -\alpha \cdot \frac{R^2 \cdot \left[7 \cdot (v_x x + v_y y)^2 + R^2 \cdot \left((v_x + \omega \cdot y)^2 + (v_y - \omega \cdot x)^2 \right) \right]}{4 \cdot c^2 \cdot r^5} \quad \textbf{(A.1.6)}$$

$$\Delta U_7 = -\alpha \cdot \frac{R^2 \cdot \left(3 \cdot (v_x^2 + v_y^2) - 2 \cdot \omega^2 \cdot r^2 + 4 \cdot \omega \cdot (v_Y x - v_x y) - 5 \cdot (a_x x + a_y y) \right)}{12 \cdot c^2 \cdot r^3} \quad \textbf{(A.1.7)}$$

$$\Delta U_8 = -\alpha \cdot \frac{R^2 \cdot \left[5 \cdot \left(1 + \frac{63 \cdot R^2}{20 \cdot r^2} \right) \cdot (v_x x + v_y y)^2 + R^2 \cdot \left(1 + \frac{5 \cdot R^2}{4 \cdot r^2} \right) \cdot \left((v_x + \omega \cdot y)^2 + (v_y - \omega \cdot x)^2 \right) \right]}{4 \cdot c^2 \cdot r^5} \quad \textbf{(A.1.8)}$$

$$\Delta U_9 = \alpha \cdot \frac{R^2 \cdot \left[\left(1 + \frac{7 \cdot R^2}{4 \cdot r^2} \right) \cdot (v_x{}^2 + v_y{}^2 - a_x x - a_y y) - \frac{R^2}{r^2} \cdot \left((v_x + \omega \cdot y)^2 + (v_y - \omega \cdot x)^2 \right) \right]}{4 \cdot c^2 \cdot r^3} \quad \textbf{(A.1.9)}$$