Computational Visualization of Retardation Effects on Observed Particle Distributions

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Abstract—**The retardation equation for straight line signal propagation is solved analytically for the two-dimensional case and used to plot the retarded positions graphically for a variety of particle distributions moving relatively to the observer. The results show that in case of a finite signal propagation speed the observed particle distribution must appear as expanded/compressed if it moves towards/away from the observer. The apparent change of scale of the distribution goes hereby along with an inverse change in particle density.**

Index Terms—**signal propagation, retardation equation, particle distribution functions**

I. Introduction

The finite propagation speed of signals in general is a well known phenomenon not only for physicists but also for laymen. Everybody knows that we can calculate the approximate distance of a thunderstorm from the time difference between the lightning flash and the rumble of thunder and the speed of sound.. Everybody also intuitively understands that we see the stars in the sky not as they are today but as they were in the past corresponding to their distance and the speed of light. What is not so obvious is the fact that in case of moving objects, this retardation effect also makes us see them in a different position than they are at the moment of observation. Not only this, but since any distribution of particles has a finite extension, its apparent configuration will in general appear changed as well due to the different distances of its parts to the observer leading to a time-layered picture.

The concept of retarded interactions due to finite signal propagation speed is of course nothing new in theoretical physics, having been developed already in the early to mid 19th century in the context of classical electrodynamics by Gauss, Riemann, Weber, Lorenz and Maxwell, and culminating later on in the derivation of the retarded electromagnetic potential of a moving point charge by Liénard and Wiechert.. However, further implications for fundamental aspects of physics have only been studied in connection with the theory of Special Relativity.. For instance, Lampa [1], Terrel [2], Penrose [3] , Weisskopf [4] Boas [5], Müller and Boblest [6] have discussed the visual appearance of fast moving objects for some regular shapes like rods, cubes or spheres, while Deissler [7] developed an algorithm applicable to arbitrarily shaped objects.. However, whilst these works have included the retardation effect into the theory, they did not consider it on its own but only in combination with the relativistic length contraction effect, which makes it didactically difficult to unambiguously illustrate the implications of the retardation effect. And for problems where one just wants to derive the actual (i.e. instantaneous) positions of a particle distribution in the observer's reference frame from the observed retarded positions, those results would not be applicable anyway.

Furthermore, all these previous works consider the retardation effect to be merely an apparent optical one as they are only concerned about the resultant projection of the object's shape into the observer's plane. This circumstance suppresses crucial features of the full retardation effect and has thus not only has led to conclusions in those papers (like the claimed 'rotation' effect) that are, as the result of the present paper will prove, at least misleading, but it also ignores the wider physical implications. It is well known for instance that retardation crucially affects the electrostatic interaction between moving particles (Liénard-Wiechert potential). In the latter context, Aguirregabiria et al. [8] derived actually the shape of a relativistically moving sphere in real space rather than as a projection, but still considered only its outline by doing an appropriate transformation of the equation of a sphere, which however ignores the fact that matter is not distributed truly continuously but consists of individual particles.

In contrast to those above mentioned works in Special Relativity (where the apparent shape of objects in the observer's reference frame was compared to their rest-frame shape), the present paper only studies the transformation from the instantaneous (actual) to the retarded (observed) particle positions in the observer's reference frame. An additional Lorentz transformation to the rest frame of the particle distribution and thus a corresponding relativistic length contraction is therefore not applicable here. (as much as it is not applicable for instance in the derivation of the retarded potentials in electrodynamics).
Furthermore, the present paper takes a novel approach by first deriving an algebraic solution to the retardation equation transforming the given instantaneous to the retarded coordinates, and then uses this to do a point for point transformation for each particle of a given discrete particle distribution (linear, square array, circular; in the latter case a

rotational velocity is considered as well)). In this way, the full information regarding the retardation effect is retained, unlike previous works showing only projections of outlines of the resulting distributions.

II. Calculation of retarded positions

Because we are interested in the retarded positions as they appear to the observer, we have to consider the situation in the latter's reference frame. And assuming that the signal propagation speed c is constant throughout space and furthermore independent of the velocity of the emitting particle, the emission time t' and observation time t_0 are related to the corresponding retarded distance d' between the points of emission and absorption through

$$d' = c \cdot (t_0 - t') \tag{1}$$

(note that the symbol c can indicate any kind of signal propagation here, not just the speed of light; it could for instance be the speed of sound if the observer is at rest in the medium (so that c does not depend on the velocity of the emitter as required above).

For the present study (which should only demonstrate the fundamental effects of the finite signal speed on observed particle distributions) we can restrict ourselves to a 2-dimensional scenario. As the algebraic results are intended to be used directly for numerical/graphical computations, it is furthermore beneficial if we formulate the equations in terms of the corresponding Cartesian x, y coordinates. We thus can write

$$d' = \sqrt{(x_0 - x')^2 + (y_0 - y')^2} \tag{2}$$

where (x_0, y_0) are the (fixed) coordinates of the observation point..
In order to simplify a comparison of the results, we make now the convention

$$t'(x' = 0, y' = 0) = 0 \tag{3}$$

i.e. a signal from the origin is emitted at $t' = 0$ (and thus observed at $t_0 = \sqrt{x0^2 + y_0^2} / c$.).

II.1 Constant Uniform Velocity

In case of a constant velocity (v_x, v_y) of the whole particle system, a particle at coordinate (x, y) at time $t' = 0$ will have the coordinates (x', y') at time t'

$$x' = x + v_x \cdot t' \tag{4}$$

$$y' = y + v_y \cdot t' \tag{5}$$

To clarify the meaning of the variables again, x_0, y_0, t_0 is the fixed observer location and time (assuming $t_0 = \sqrt{x0^2 + y_0^2} / c$), x, y designates the (given) source particle location at time $t' = 0$ (the emission time for a particle at the origin to be observed at time t_0 at x_0, y_0), whereas the unknowns x', y' and t' designate the retarded location and time of that particle required to be observed at time t_0 at x_0, y_0 as well). The velocity components v_x, v_y should in principle also be retarded quantities, but as they are assumed constant here, we can afford to write them unprimed.

Inserting (4) and (5) into (2) and squaring the resulting equation yields a quadratic equation for t' which has the solution

$$t' = t_0' \mp \sqrt{t_0'^2 + \frac{x^2 + y^2 - 2 \cdot x_0 \cdot x - 2 \cdot y_0 \cdot y}{c^2 - v_x^2 - v_y^2}} \tag{6}$$

with

$$t_0' = \frac{c^2 \cdot t_0 - v_x \cdot (x_0 - x) - v_y \cdot (y_0 - y)}{c^2 - v_x^2 - v_y^2} \tag{7}$$

(the variable t_0' does mot have a particular meaning here; it is only introduced in order be able to write (6) in a more compact format so that it fits into the 2-column layout, and as it reduces to t_0 for a particle at rest, this particular notation was chosen).

With this definition, the negative sign in front of the square root in (6) applies for $c^2 - v_x^2 - v_y^2 > 0$ and the positive sign for $c^2 - v_x^2 - v_y^2 < 0$ (note again that c denotes here a signal propagation speed in general, not just the speed of light, so both cases are in general formally possible on this basis, although the latter case will not be discussed here)..This corresponds to the retarded solution of (2), whereas the opposite sign in the two cases corresponds to the advanced solution (which is not considered here as it would violate causality).

Equation (6) yields thus the retarded time (i.e. time of emission of the signal) of a particle which is at position (x, y) at time $t' = 0$ and moving with velocity (v_x, v_y) for an observation at position (x_0, y_0). at time t_0..
The corresponding retarded positions are then simply obtained from (4) and (5).
Note that the above equations are exact for all velocities as they involve no approximation (provided the velocities are constant between times $t' = 0$ and $t' = t_0$).

II.2 Constant Uniform Velocity + Rotation

If the distribution is located on a circle with radius r and rotates around its centre with angular velocity ω, with the whole distribution additionally having a constant velocity, (4) and (5) have to be replaced by

$$x' = r \cdot \cos(\varphi + \omega \cdot t') + v_x \cdot t' \tag{8}$$

$$y' = r \cdot \sin(\varphi + \omega \cdot t') + v_y \cdot t' \tag{9}$$

Assuming $\omega \cdot t' << 1$ (which effectively means that the rotational velocity $v_r << c$ as $v_r = \omega \cdot r$ and $|t'| \approx r/c$ because of convention (3)) and applying the addition theorem to the trigonometric functions and using a Taylor expansion of $\cos(\omega \cdot t')$ and $\sin(\omega \cdot t')$ up to second order in $\omega \cdot t'$ we obtain analogously to (6) and (7)

$$t' = t_0' \mp \sqrt{t_0'^2 + \frac{r^2 - 2 \cdot r \cdot (x_0 \cdot \cos(\varphi) + y_0 \cdot \sin(\varphi))}{c^2 - v_x^2 - v_y^2 - r \cdot \omega \cdot v_{x,y}}} \tag{10}$$

with

$$t_0' = \frac{c^2 \cdot t_0 - v_x \cdot x_1 - v_y \cdot y_1 - r \cdot \omega \cdot R_0}{c^2 - v_x^2 - v_y^2 - r \cdot \omega \cdot v_{x,y}} \tag{11}$$

where

$$x_1 = x_0 - r \cdot \cos(\varphi) \tag{12}$$

$$y_1 = y_0 - r \cdot \sin(\varphi) \tag{13}$$

$$R_0 = y_0 \cdot \cos(\varphi) - x_0 \cdot \sin(\varphi) \tag{14}$$

$$v_{x,y} = (2 \cdot v_y - x_0 \cdot \omega) \cdot \cos(\varphi) - (2 \cdot v_x - y_0 \cdot \omega) \cdot \sin(\varphi) \tag{15}$$

(the variables defined in (12)-(15) have no specific meaning but are only placeholders in order to enable a more compact form of (10) and (11).)

Analogously to the case of a constant velocity (to which (8)-(11) are easily seen to reduce for $\omega = 0$) the negative sign in front of the square root in (11) applies if the denominator of $t_0' > 0$ and the positive sign if it is < 0.).

III. NUMERICAL RESULTS

The retarded positions of different particle distributions have been numerically evaluated on the basis of the equations in Sect.2. By the nature of convention (3), all distributions remain centered around the origin (a particle at the origin is unaffected by the retardation) so the changes of the shape of the distributions as a result of the finite signal propagation speed can be compared directly (effectively, we always choose an observation time so that the observer sees the centre of the distribution at the origin).. Given the nature of the problem, the results depend obviously crucially on the observer location as well as the velocity vector(s) of the particle distribution with regard to the observer, so this information is added to all figure captions in order to avoid confusion.

All graphical plots were produced with Mathematica V9 on the basis of the equations derived in this paper,

III.1 One-Dimensional case

We consider here just a number of particles equidistantly distributed on the x-axis, with the center of the distribution observed at the origin. The below graphs shows the result for the cases a) instantaneous particle positions (circles), b) retarded positions for distribution moving towards the observer with velocity $v_x = 0.5c$ (squares, offset from the x-axis for clarity), c) retarded positions for distribution moving away from the observer with velocity $v_x = -0.5c$ (triangles, offset from the x-axis for clarity). . In either case, the observer has been assumed at $x_0 = 3,\ y_0 = 0$.

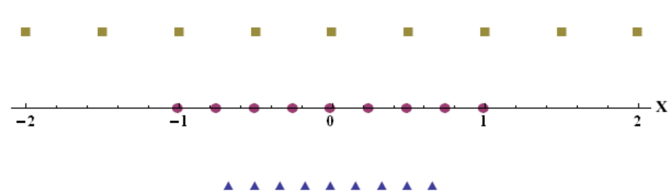

Fig.1 Effect of retardation on moving linear particle distribution

As the numerical results show, the observed retarded distribution appears expanded or compressed compared to the instantaneous one depending on whether it is approaching or receding with regard to the observer. It is easy to show (see Appendix A.1. that the retarded distribution is in this case in fact given by the equation

$$x' = \frac{x}{1 - v_x / c} \qquad (16)$$

(it can also be shown from the general equations (4)-(7) above, but this is algebraically not exactly straightforward).

This means obviously that not only are the endpoints between the two distributions expanded/contracted but also the distance between the individual particles and thus the particle density i,e

$$\rho'(v_x) = \rho_0 \cdot (1 - \frac{v_x}{c}) \qquad (17)$$

where ρ_0 is the instantaneous particle density.
From (16) and (17) it follows immediately that

$$\rho' \cdot dx' = \rho_0 \cdot dx \qquad (18)$$

which reflects the particle conservation law (the total number of particles is obviously unchanged by the apparent expansion or compression of the scale of the distribution resulting from the finite propagation speed of the signal; this applies to any sub-volume as well).

III.2 Two-Dimensional case

In contrast to the previous section we assume now a square grid of points in the region x=-1 to 1 and y=-1 to 1 at intervals of $\Delta x, \Delta y$ =0.25 for the instantaneous distribution (Fig.2)

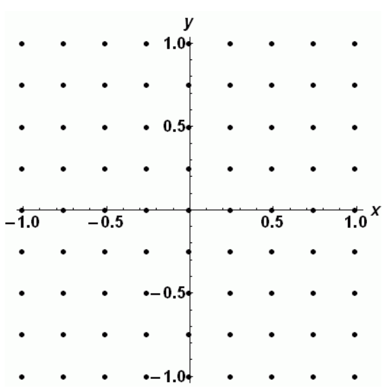

Fig.2 Instantaneous positions for square array

The following two plots show the retarded positions if the whole distribution is moving, analogously to Sect. II.1, with speed $v_x = 0.5c$ and $v_x = -0.5c$ towards and away from the observer (again assumed at $x_0 = 3,\ y_0 = 0$) with $v_y = 0$

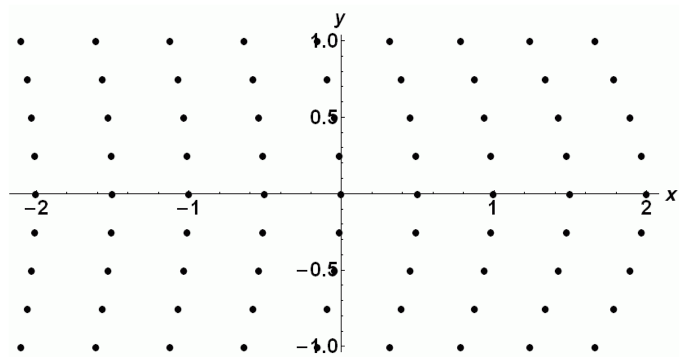

Fig.3 Retarded distribution for $v_x = 0.5c$, $v_y = 0$ for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0 / c$

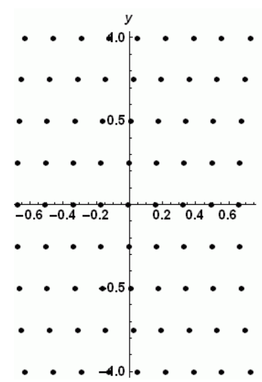

Fig.4 Retarded distribution for $v_x = -0.5c$, $v_y = 0$ for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0 / c$

Note that in all three plots above the scales of the units are identical, so the apparent sizes of the particle distributions can be compared directly.

It is obvious that the two-dimensional result for motion parallel to the x-axis is just a generalization of the one-dimensional case, with the expansion/contraction of the distribution being less pronounced for parallels along the x-axis according to the projection of the particles velocity vector on the line of sight. The latter circumstance also means that the two dimensional distribution depends on the location of the observer on the x-axis. For significantly more distant observation points, the velocity vectors of all particles deviate much less from the line of sight, so the expansion/ compression is much more uniform across the whole distribution, essentially approaching the one-dimensional case as given by (17) and (18). The below plot shows for instance the solution for $v_x = 0.5c$, so analogously to Fig.3, but now with the observer at $x_0 = 10$

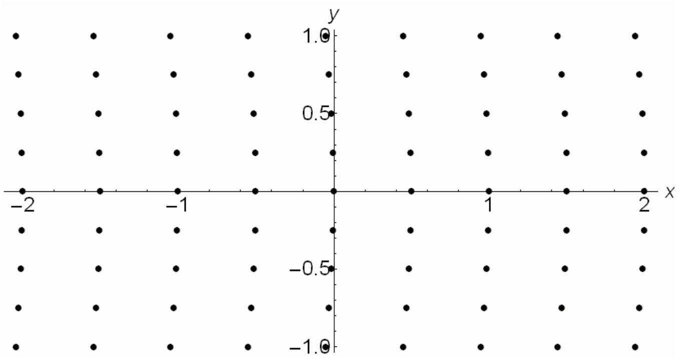

Fig.5 Retarded distribution for $v_x = 0.5c$, $v_y = 0$ for an observer position and time $x_0 = 10$, $y_0 = 0$, $t_0 = x_0/c$

A different picture arises if the distribution is moving along thee y-axis i..e. perpendicular to the line of sight of the observer. For $v_x = 0, v_y = 0.5c$, we obtain the following distribution

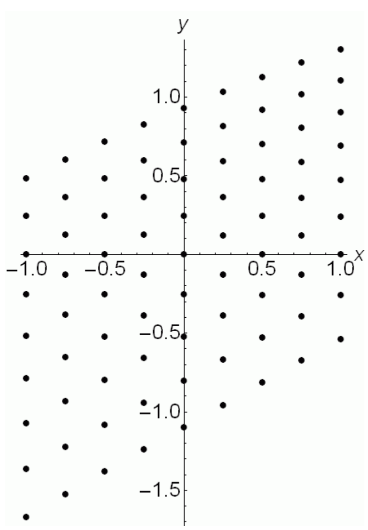

Fig.6 Retarded distribution for $v_x = 0$, $v_y = 0.5c$ for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0/c$

In this case (with the observer position again located at $x_0 = 3,\ y_0 = 0$) the distribution is barely expanded or contracted (only slightly in the y-direction), but merely sheared due to the retardation effect (which results in particles at positive x being observed at a later time compared to a particle at the origin and particles with negative x at an earlier time.). For larger distances of the observer the expansion/compression is even smaller and indeed disappears completely for $x_0 \to \infty$.

For some reason most of the references cited in this paper (and also some educational textbooks like that of Greiner [9]) describe the appearance of a transversely moving cube as a rotated one, but even though they refer only to the projection of this distribution (and the projection of a sheared and rotated distribution would be the same here if relativistic length contraction is additionally taken into account), it is at least misleading to suggest the distribution would be rotated in space (this was pointed out already in 1972 by Matthews and Lakshmanan [10]). As the retarded position must lie on the particle trajectory, a rotation is obviously physically impossible if all parts of the object move on a straight line (which is the assumption here).

In case of the velocity vector of the distribution being parallel to the x,y diagonal (i.,e. for $v_x = 0.5c$, $v_y = 0.5c$ we have essentially a combination of Figs, 3 and 6

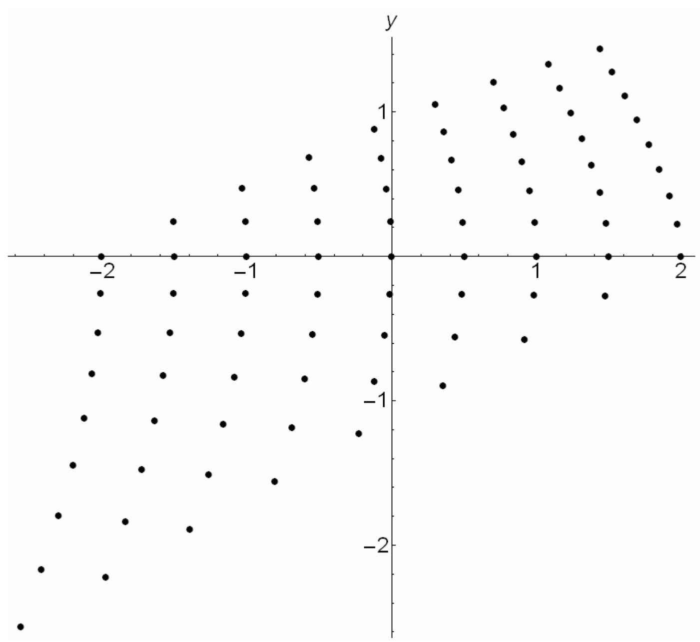

Fig.7 Retarded distribution for $v_x = 0.5c$, $v_y = 0.5c$ for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0/c$

III.3 Rotating Distributions

For the case of a rotating (but otherwise stationary) ring, we obtain on the basis of the equations in Sect. II.2 the following retarded distribution for $c = 1, r = 1$ and $\omega = 0.5/$

$\omega = -0.5$ respectively (i..e. $v_r = 0.5$), with the observer again located at $x_0 = 3,\ y_0 = 0$

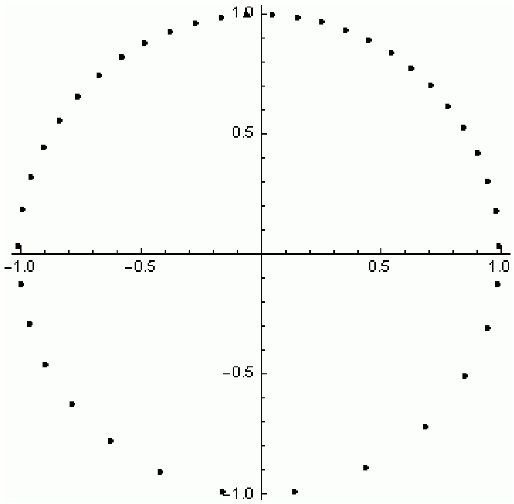

Fig.8 Retarded distribution for ring rotating with $v_r = 0.5c$ anti-clockwise for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0/c$

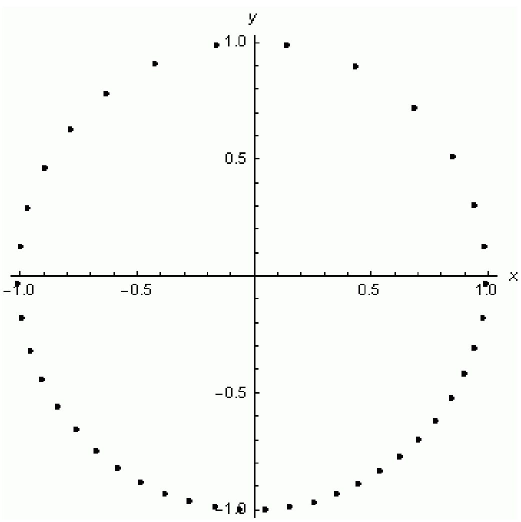

Fig.9 Retarded distribution for ring rotating with $v_r = 0.5c$ clockwise for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0/c$

Since all particles are confined to moving on a circular path (in anti-clockwise direction for this example), there is in this case no distortion of the shape as a whole but merely of the particle density on the ring itself. The interpretation is straightforward here on the basis of the previous sections: parts of the circle that are receding from the observer due to its rotation are compressed (i.e. have a higher density) parts that are approaching are expanded (i..e have lower density).

(it should be noted here that unlike for the uniform velocity results shown earlier, the plots for the rotating distribution are not quite exact here as the corresponding Eq. (11) is only accurate up to orders of v_r^2/c^2; these plots should therefore only be considered as qualitative illustrations of the effect,)

If we additionally have the rotating ring move as a whole with a uniform velocity in the $\pm x$ direction, we obtain a combination of Fig.8 with the expansion/compression effect shown in Figs.(3) and (4)

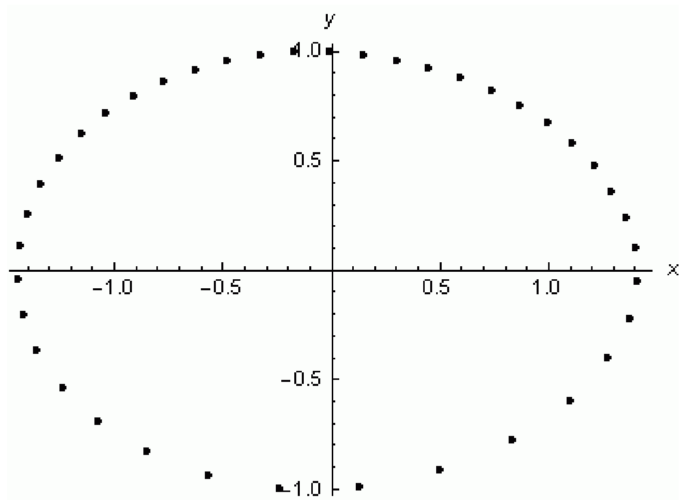

Fig.10 Retarded distribution for ring rotating with $v_r = 0.3c$ anti-clockwise and moving with $v_x = 0.3c$, $v_y = 0$ for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0/c$

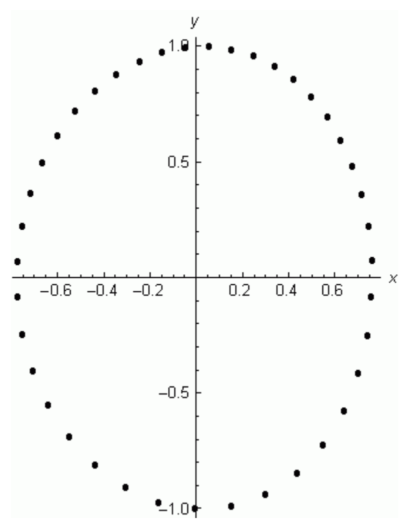

Fig.11 Retarded distribution for ring rotating with $v_r = 0.3c$ anti-clockwise and moving with $v_x = -0.3c$, $v_y = 0$ for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0/c$

whereas for the rotating distribution moving additionally in the +y direction we obtain a combination of Fig.(8) with Fig.(6)

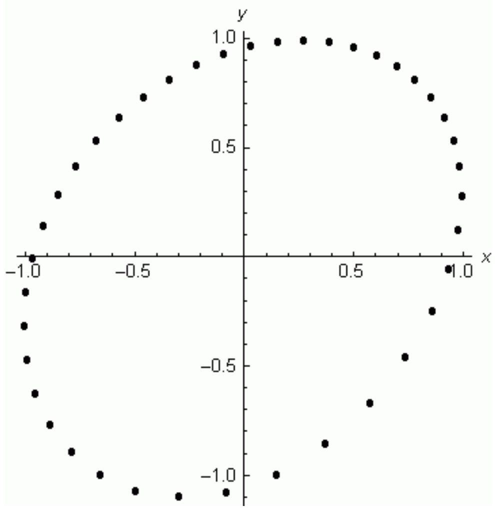

Fig.12 Retarded distribution for ring rotating with $v_r = 0.3c$ anti-clockwise and moving with $v_x = 0$, $v_y = 0.3c$ for an observer position and time $x_0 = 3$, $y_0 = 0$, $t_0 = x_0/c$

Again, for all these plots the scales of the units are identical, so the particle distributions are directly comparable.

III.3 High Velocity/ Close Distance Effects

A strange effect appears if the distribution approaches the observer so fast that some parts of it have already passed the observer when the midpoint is still seen at the origin.. In this case the part of the distribution still approaching the observer will appear as expanded but the part having passed already as compressed. The distance of the observer to the distribution (which in the above examples had only a minor effect on the results) plays obviously a crucial role in this case. The below plot shows how the unit square array of Fig (1) looks for an observer at $x_0 = 1.2$ if it is moving towards him with $v_x = 0.8c$

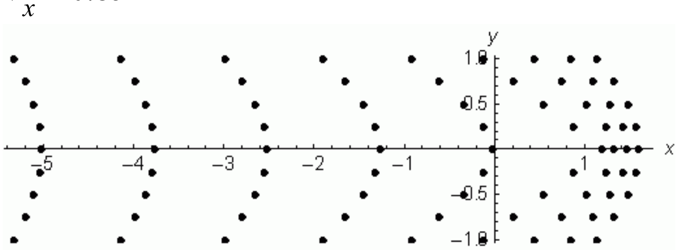

Fig.13 Retarded distribution for $v_x = 0.8c$, $v_y = 0$ for an observer position and time $x_0 = 1.2$, $y_0 = 0$, $t_0 = x_0/c$

With the central particle (by definition) still at the origin at this moment, some particles have already passed the observer and thus this 'downstream' distribution appears to have a higher density compared to the instantaneous case, whereas the 'upstream' distribution has a lower density (see also Figs. (3) and (4)).

IV. CONCLUSIONS AND OUTLOOK

The present work has obtained detailed graphical results that illustrate the effect of a finite signal propagation speed on the perceived spatial configuration of a moving particle distribution. In contrast to other results discussing only the photographic projection of the resulting shape of objects, the full spatial information is preserved here (albeit only for a 2D-scenario in order to be graphically representable) and most importantly, the point-for-point transformation for the individual particles has additionally revealed information about the apparent density changes within the distribution. Being able to fully visualize the effects of a finite signal propagation speed in this sense should be helping teachers as well as students to better understand the intricacies introduced by the time layering in the signal retardation for moving particle distributions and avoid misleading conclusions based only on their projections and/or outlines. It should also be realized that even though the retarded distributions are only apparent (i.e. observer dependent) ones and thus do not imply any internal changes in the objects themselves , they are lastly the only physically relevant ones as the instantaneous particle distribution can by definition never be seen directly for finite signal propagation speeds. This may thus not only require a corresponding correction in the interpretation of observational data, but also in the modelling of the physical interaction of objects in motion, as some simplifying assumptions (like the shell theorem for spherical distributions) may not be applicable anymore.

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APPENDIX A.1

ALGEBRAIC DERIVATION OF RETARDED POSITIONS FOR A ONE-DIMENSIONAL ARRAY

In order to better understand the various numerical results resulting from the full 2D-equations in Sect.II and discussed in Sect.III, it is instructive to derive the retarded positions immediately algebraically for the simple case of a one dimensional problem. i.e. for the distribution, its velocity vector and the observer all located on the x-axis. The argument is similar to that given in some textbooks like Griffiths [11]. (this could also be derived as a special case of the general equations in Sect.II, but it is algebraically quite involved and anything but straightforward)..

If we have a number of point charges at locations x within a finite length L and this whole configuration moving with speed v with regard to the observation point x_0 (assumed > x) on the same line. The retardation condition in this case is

$$x_0 - x' = c \cdot (t - t') \qquad \text{(A.1.1)}$$

where x', t' is the point and time of emission and x_0, t the point and time of detection. If we select the zero point of the time variable such that $x_0 = ct$, the retardation condition is thus

$$x' = c \cdot t' \qquad \text{(A.1.2)}$$

For the location of a uniformly moving charge, we have furthermore the constraint

$$x' = x + v \cdot t' \qquad \text{(A.1.3)}$$

where x is the position of the charge at $t' = 0$., from which we get immediately

$$x' = \frac{x}{1 - v/c} \qquad \text{(A.1.4)}$$

This means that for $v > 0$ (array approaching the observer) the scale of the distribution appears as expanded (x'>x) and for $v < 0$ (array receding from the observer) it appears as compressed., as confirmed by the numerical results based on the more general equations derived in Sect.II.