Normalization of Density Distribution Functions and the Calculation of Retarded Potentials and Fields

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***Abstract*—The usual derivation of retarded potentials in textbooks and other educational resources is reviewed and analyzed in the light of rigorous definitions for the scale transformation of the density distribution function of a system of particles. It is shown that scale changes of particle distribution functions must in general result in corresponding inverse changes of the particle density such as to preserve the total number of particles. Applied to the problem of the calculation of the electric potential of moving charges, the well known Liénard-Wiechert potential and the associated expressions for the electric and magnetic fields of moving charges prove to be erroneous in this sense as they ignore the density change and thus artificially increase or decrease the total number of charges depending on the change of the scale transformation with aspect angle, leading to a divergent behavior for particle velocities approaching the speed of light. A correct derivation preserving the normalization of the distribution function displays indeed no such divergent properties but the solution remains finite throughout. The present work also corrects an error in the usual calculation of the electric field due to the acceleration term.**

***Index Terms*—distribution functions, scale transformations, classical electrodynamics, Liénard-Wiechert potential, relativistic electrodynamics**

1. Introduction

Although expressions for the velocity and acceleration dependence of electromagnetic forces and the concept of retarded interactions were already developed by Gauss, Riemann, Weber, Lorenz and Maxwell in the early to mid 19th century, the notion of retarded potentials only came into general use after Hertz's experiments on electromagnetic waves and the discovery of the electron by Thomson towards the end of the century. This laid the basis for a kinematic consideration of the motion of charged particles and the resulting retarded potentials and fields. In this respect, Liénard [1] and Wiechert [2] were the first who, independently of each other, obtained the retarded electric and magnetic potentials of a moving point charge (which are consequently named Liénard-Wiechert potentials). Their solution, which is the basis of the corresponding derivations in all textbooks about electrodynamics (e.g. Jackson [3], Griffiths [4] or Feynman [5]), involves a spatial integration over the retarded density distribution i.e. a distribution which has undergone a scale transformation (being expanded or compressed compared to the unretarded distribution). However, all these derivations imply that the density is unaffected by this scale transformation, which means the transformed density distribution is not normalized anymore (or in terms of the physical interpretation here, the total charge is not conserved). In some textbooks, this lack of normalization for the retarded distribution is not merely made implicitly, but is explicitly taken for granted, as it is apparently seen as a natural mathematical consequence of the scale change of the distribution caused by the retardation (c.f. for instance Sect.10.3.1 in Griffiths [4]). However, this argument completely ignores the fact that the overall scale change of the particle distribution must go along with a corresponding scale change of the inter-particle distance i.e. the local density. In this sense, sect. 2 of the present paper first formally examines the behavior of density distribution functions under arbitrary (but particle conserving) spatial transformation in general and linear scale transformations in particular. On this basis, sect. 3 formulates then the expression for the scalar potential associated with a spatially transformed density distribution in general and applies this result to the case of a moving point charge. In sect. 4 (with the detailed derivation in Appendix A.1) the correct expression for the electric field of a moving point charge is derived from the so obtained scalar and vector potential. The result differs from the usual expression found in the literature not only because of the normalization issue mentioned above, but also due to a further mathematical mistake in the in the derivation of the Liénard-Wiechert fields when calculating the contribution of the acceleration term in the vector potential.

2. Normalization of density distribution functions

The concept of density distribution functions is a very important one in theoretical physics, as it enables the calculation of certain macroscopic physical quantities of many-particle systems in a closed way (something that would be difficult or impossible by considering individual particle kinematics). The density distribution function is therefore nothing more than a probability distribution function and as such is subject to certain constraints set by the physical state of the system (only with these constraints is it mathematically legitimate in the first place to approximate an actually discrete distribution function by a continuous one; see Appendix A.4 for a more detailed discussion of this issue). In particular is it required that the integral of the density distribution function over space must yield the total number of particles, and

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furthermore that this number is independent of time if the system is closed (that is. no particles are gained or lost), i.e.. for an ensemble of particles with the (average) volume density $\rho(\mathbf{r},t)$ we have therefore the constraint

$$\int d^3\mathbf{r}\,\rho(\mathbf{r},t) = N \qquad (1)$$

where N is the total number of particles, and the integration is performed over all $\mathbf{R}^3$ (or the volume elements for which $\rho(\mathbf{r},t) > 0$).

We re-arrange now the spatial positions of the particles to yield a different distribution function $\rho'(\mathbf{r},t)$. Since no particles are gained or lost through a simple spatial redistribution, we must therefore also have

$$\int d^3\mathbf{r}\,\rho'(\mathbf{r},t) = N \qquad (2)$$

One can obviously relate the old and new distribution functions generally through

$$\rho'(\mathbf{r},t) = k(\mathbf{r}'(\mathbf{r},t),t)\cdot \rho(\mathbf{r}'(\mathbf{r},t),t) \qquad (3)$$

where $\mathbf{r}'(\mathbf{r},t)$ is an arbitrary transformation from the old particle coordinates $(\mathbf{r},t)$ to the new coordinates $(\mathbf{r}',t)$, and $k(\mathbf{r}'(\mathbf{r},t),t)$ is a, as yet unknown, normalizing function which ensures (2) is fulfilled.,

Inserting (3) into (2) yields then

$$\int d^3\mathbf{r}\; k(\mathbf{r}'(\mathbf{r},t),t)\cdot \rho(\mathbf{r}'(\mathbf{r},t),t) = N \qquad (4)$$

and after changing the integration variable from r to r'

$$\int d^3\mathbf{r}'\,\frac{k(\mathbf{r}'(\mathbf{r},t),t)\cdot \rho(\mathbf{r}'(\mathbf{r},t),t)}{\lambda(\mathbf{r}'(\mathbf{r},t),t)} = N \qquad (5)$$

where, according to fundamental calculus,

$$\lambda(\mathbf{r}'(\mathbf{r},t),t) = \left|\det\left(\frac{\partial \mathbf{r}'}{\partial \mathbf{r}}\right)\right| \qquad (6)$$

the absolute value of the determinant of the 'Jacobian' matrix

$$\left(\frac{\partial \mathbf{r}'}{\partial \mathbf{r}}\right) = \frac{\partial \mathbf{r}'_i}{\partial \mathbf{r}_j} \qquad (7)$$

of the first derivatives between the vector components of $\mathbf{r}$ and $\mathbf{r}'(\mathbf{r},t)$ ...

From a comparison of (5) with (1) it follows thus

$$k(\mathbf{r}'(\mathbf{r},t),t) = \lambda(\mathbf{r}'(\mathbf{r},t),t) \qquad (8)$$

(this is immediately evident if we take the integral in both equations to some upper limit and differentiate with regard to this limit)) , so

$$\rho'(\mathbf{r},t) = \lambda(\mathbf{r}'(\mathbf{r},t),t)\cdot \rho(\mathbf{r}'(\mathbf{r},t),t) \qquad (9)$$

For the special case of a simple scale transformation along, let's say, the x-axis, i.e. for

$$x'(x,t) = \lambda\cdot x \qquad (10)$$

where λ is some dimensionless scale factor, we have therefore

$$\rho'(x,t) = \lambda\cdot \rho(\lambda\cdot x,t)\,. \qquad (11)$$

In other words, distributing the same particles over a different scale changes the particle density accordingly, which is also intuitively obvious (note here that $\lambda > 1$ reduces the scale of the spatial distribution and $\lambda < 1$ expands it).

In the next section we will apply this result to the case of the evaluation of the retarded potential of a system of moving charges.

3. Calculation of retarded potentials

For a stationary density distribution $\rho(\mathbf{r})$, the electric potential at point $\mathbf{R}$ is generally given by (using Gaussian cgs – units)

$$\phi(\mathbf{R}) = q\cdot\int d^3\mathbf{r}\,\frac{\rho(\mathbf{r})}{|\mathbf{R}-\mathbf{r}|} \qquad (12)$$

with q the individual particle charge (assumed to be identical here for all particles).

If instead we have a time dependent distribution due to particle motion and assume a finite propagation speed of the potential, the total number of particles (and thus the total charge here) is preserved, but the instead of $\rho(\mathbf{r})$ we have to integrate over a different distribution $\rho'(\mathbf{r},t)$ that arises from the apparent time-layering due to the finite propagation speed, so with Eq.(9) we have (replacing the original position $\mathbf{r}$ in (12) with the retarded position $\mathbf{r}'(\mathbf{r},t)$)

$$\phi(\mathbf{R},t) = q \cdot \int d^3\mathbf{r} \frac{\lambda(\mathbf{r}'(\mathbf{r},t),t) \cdot \rho(\mathbf{r}'(\mathbf{r},t),t)}{|\mathbf{R} - \mathbf{r}'(\mathbf{r},t)|} \qquad (13)$$

(in order to apply the concept of retarded positions , we have to assume here that $\mathbf{r}'(\mathbf{r},t)$ is a single-valued function, so effectively we have to consider the positions of individual particles)…
.
Note that in (13) the density distribution is still formally written as a function of t (that is, unretarded in time), but it is a function relating to the retarded positions $\mathbf{r}'(\mathbf{r},t)$ .The latter are the only relevant aspect here for calculating the potential.

Changing the integration variable to $\mathbf{r}'$ analogously to (4),(5) gives then

$$\phi(\mathbf{R},t) = q \cdot \int d^3\mathbf{r}' \frac{\rho(\mathbf{r}'(\mathbf{r},t),t)}{|\mathbf{R} - \mathbf{r}'(\mathbf{r},t)|} \qquad (14)$$

Note that contrary to the usual derivation of the Liénard-Wiechert potential in the literature (as given originally by Liénard [1] and Wiechert [2] and repeated in many textbooks like those by Jackson [3], Griffiths [4] or Feynman [5]), there is no factor $1/\lambda$ resulting from the variable change anymore, as it cancels out with the normalizing factor for the retarded density distribution (9). So there is in fact no direct dependence of the potential on the velocity v of the charged particles. The only difference of (14) to the stationary case (12) is that the density distribution ρ reflects the retarded positions $\mathbf{r}'(\mathbf{r},t)$ rather than the instantaneous positions $\mathbf{r}$. For instance, for a localized particle at the retarded position $\mathbf{r}'(\mathbf{r},t_0) = 0$ i.e. for a density distribution

$$\rho(\mathbf{r}'(\mathbf{r},t_0),t_0) = \delta^3(\mathbf{r}'), \qquad (15)$$

(14) becomes

$$\phi(\mathbf{R},t_0) = \frac{q}{|\mathbf{R}|} \qquad (16)$$

that is the same potential a charged particle at rest at the origin creates.

In general, the potential depends obviously on the retarded positions $\mathbf{r}'(\mathbf{r},t)$ of all the charges. Once these have been determined (i.e. once the retarded distribution $\rho(\mathbf{r}'(\mathbf{r},t),t)$ is known) the potential is derived in the same way as that of an identical static distribution. (i.e. as in (12)), although of course the retarded positions $\mathbf{r}'(\mathbf{r},t)$ depend in general on the observation point $\mathbf{R}$, so they have to be re-evaluated for each $\mathbf{R}$ and t .

In order to illustrate the effect that the retardation has on the apparent charge density distribution and the resulting potential, we shall consider here a one dimensional scenario with a number of point charges at locations x within a finite length L and this whole configuration moving with speed v with regard to the observation point X (assumed $>x$) on the same line. The retardation condition in this case is

$$X - x' = c \cdot (t - t') \qquad (17)$$

where x', t' is the point and time of emission and X, t the point and time of detection. If we select the zero point of the time variable such that $X = ct$, the retardation condition is thus

$$x' = c \cdot t' \qquad (18)$$

For the location of a uniformly moving charge, we have furthermore the constraint

$$x' = x + v \cdot t' \qquad (19)$$

where x is the position of the charge at $t' = 0$., from which we get immediately

$$x' = \frac{x}{1 - v/c} \qquad (20)$$

In Fig.1 below, this result is shown graphically for a number of charges between $x = -L/2$ and $x = +L/2$, at the top for the charges at rest, and at the bottom according to (20) for the whole configuration moving with speed $v = 0.5 \cdot c$ towards the observation point .

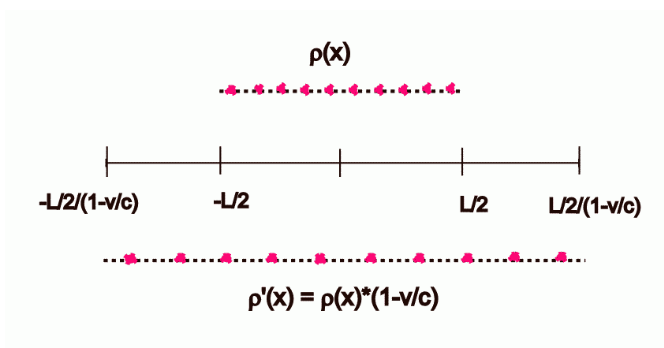

Fig.1 : Retarded vs. unretarded density distribution

.The expanded scale of the retarded distribution is due to the time layering of the picture caused by the different distances of the charges to the observation point X, which means that (assuming all charges are moving towards the observer) a signal from a distance closer than the origin has to be emitted at a later time t' in order to be observed at time t at point X, whereas a signal from a distance further than the origin has to be emitted at an earlier time. The retarded distribution is therefore spread over a corresponding range of retarded times. This of course is nothing new as such, and the derivation of the retarded potentials in the literature (following on from the work of Liénard [1] and Wiechert [2]) is indeed based on this circumstance, but it has been apparently overlooked that the expanded scale of the retarded distribution (by a factor 2 in this example) must go along with a corresponding decrease of its density, if the total charge should be preserved (as already shown formally in sect. 2 for particles redistributed in space in general.). Trying to keep the charge density in the bottom section in the diagram identical to that at the top whilst maintaining the expanded scale would be equivalent to doubling the number of charges for the retarded distribution (or alternatively doubling the value of the elementary unit charge). The mistake made in the usual derivation of the Liénard-Wiechert potential is to assume (c.f. for instance Sect. 10.3.1 in Griffiths [4]; also Aguirregabiria et al. [6] for the case of a spherical charge distribution) the local density would be unaffected by the overall scale change of the particle distribution. (see Appendix A.4. for a more detailed discussion of this issue).

Of course, for a spatially expanded distribution as in this case, the apparent change of the spatial scale due to the retardation will result in a velocity dependent potential observed at a point X, but this velocity dependence disappears if the scale of the charge distribution becomes infinitesimally small. For instance, if we consider just the two outermost charges in Fig.1 i.e. if

$$\rho(x) = \delta(x - L/2) + \delta(x + L/2) \tag{21}$$

and thus

$$\rho(x') = \delta\left(x' - \frac{L}{2\cdot(1 - v/c)}\right) + \delta\left(x' + \frac{L}{2\cdot(1 - v/c)}\right) \tag{22}$$

so

$$\phi(X) = q \cdot \int d^3x' \frac{\rho(x')}{|X - x'|} =$$

$$= \frac{q}{X - \frac{L}{2\cdot(1 - v/c)}} + \frac{q}{X + \frac{L}{2\cdot(1 - v/c)}} \tag{23}$$

If we let now $L \to 0$ (i.e. joining the two particles with charge q into one particle with charge 2q at the origin), we obtain

$$\phi(X) = \frac{2q}{X} \tag{24}$$

that is the classic Coulomb potential of a charge 2q located at the origin, regardless of the speed of that charge with regard to the observer at point X.

4. Electric field of moving charge

Whilst potentials are a useful mathematical concept in physics in general, the observationally relevant quantity is only the associated field that causes the local action on any particles. On the basis of Maxwell's equations, the electric field is in this sense obtained from the equation (again in Gaussian cgs-units)

$$\mathbf{E}(\mathbf{R},t) = -\nabla\phi(\mathbf{R},t) - \frac{1}{c}\cdot\frac{\partial \mathbf{A}(\mathbf{R},t)}{\partial t} \tag{25}$$

where, analogously to the scalar potential (14),

$$\mathbf{A}(\mathbf{R},t) = \frac{q}{c}\cdot\int d^3\mathbf{r}' \frac{\mathbf{v}(\mathbf{r}'(\mathbf{r},t))\cdot\rho(\mathbf{r}'(\mathbf{r},t),t)}{|\mathbf{R} - \mathbf{r}'(\mathbf{r},t)|} \tag{26}$$

is the vector potential created at point $\mathbf{R},t$ due to motion of charges at their (retarded) positions $\mathbf{r}'(\mathbf{r},t)$. (as in the previous sections, all primed variables indicate here and in the following retarded quantities).

If we want to calculate the field of an individual charge, we have to set the density distribution

.

$$\rho(\mathbf{r}'(\mathbf{r},t),t) = \delta^3(\mathbf{r}' - \mathbf{s}(t'(t))) \tag{27}$$

where $\mathbf{s}(t'(t))$ is the path vector of the particle at the retarded time $t'(t)$. With this, the potential and vector potential become (from (14) and (26))

$$\phi(\mathbf{R},t) = \frac{q}{|\mathbf{d}'(\mathbf{R},t)|} \tag{28}$$

.

$$\mathbf{A}(\mathbf{R},t) = \frac{\mathbf{v}(\mathbf{s}(t'(t)))}{c}\cdot\frac{q}{|\mathbf{d}'(\mathbf{R},t)|} \tag{29}$$

where

$$\mathbf{d}'(\mathbf{R},t) = \mathbf{R} - \mathbf{s}(t'(t)) \tag{30}$$

It is straightforward to show (see Appendix A.1) that in this case

$$\nabla \phi(\mathbf{R},t) = \frac{-q}{\lambda(\mathbf{R},t)} \cdot \frac{\mathbf{d}'(\mathbf{R},t)}{|\mathbf{d}'(\mathbf{R},t)|^3} \qquad (31)$$

and

$$\frac{1}{c} \cdot \frac{\partial \mathbf{A}(\mathbf{R},t)}{\partial t} = \frac{q}{c^2} \cdot \left[\frac{\mathbf{a}'}{\cdot |\mathbf{d}'(\mathbf{R},t)|} + \frac{1}{\lambda(\mathbf{R},t) \cdot} \frac{\mathbf{v}' \cdot v' \cdot \cos(\theta)}{|\mathbf{d}'(\mathbf{R},t)|^2} \right] \qquad (32)$$

with

$$\lambda(\mathbf{R},t) = 1 - \frac{v'}{c} . \cos(\theta) \qquad (33)$$

where θ is the angle between the velocity vector $\mathbf{v}'$ and the relative vector $\mathbf{d}'(\mathbf{R},t)$ from the retarded position of the particle .to the observation point
Furthermore, $\mathbf{a}' = \partial \mathbf{v}' / \partial t'$ in (32).

With (25), the electric field thus becomes

$$\mathbf{E}(\mathbf{R},t) = \frac{q}{\lambda(\mathbf{R},t)} \cdot \frac{\mathbf{d}'(\mathbf{R},t)}{|\mathbf{d}'(\mathbf{R},t)|^3} - \frac{q \cdot v' \cdot \cos(\theta)}{c^2 \cdot \lambda(\mathbf{R},t)} \cdot \frac{\mathbf{v}'}{|\mathbf{d}'(\mathbf{R},t)|^2} - \frac{q}{c^2} \cdot \frac{\mathbf{a}'}{|\mathbf{d}'(\mathbf{R},t)|} \qquad (34)$$

The first term in this expression is solely due to the potential gradient and constitutes a field vector pointing from the retarded position of the charge to the observation point, whereas the second and third terms are solely due to the induction term (32) and represent field vectors pointing anti-parallel to the retarded velocity and the retarded acceleration respectively. Both of the latter are only of second order (~ v'^2/c^2 and $\sim a'/c^2$), with the velocity dependent term vanishing altogether if the velocity vector is normal to the position vector ($\cos(\theta) = 0$).

It is furthermore of crucial importance that the acceleration term does not contain the velocity dependent factor $1/\lambda(\mathbf{R},t) = 1/(1 - \mathbf{u}' \cdot \mathbf{v}'/c)$. Otherwise, Gauss' law would not hold (see Appendix A.2). Nevertheless, in contrast to the usual result for the Liénard-Wiechert field (see Appendix A.3), the local radial flux due to the acceleration term does in fact not vanish identically here, but only as an average over the whole sphere.

One should also note that despite the factor $1/\lambda(\mathbf{R},t)$ in the first two terms in (34), neither the radial nor the transverse electric field at any point $\mathbf{R}$ diverges as v' approaches c, because by scalar multiplication with $\mathbf{u}'$ we obtain (ignoring the acceleration term)

$$E_r = \mathbf{E}(\mathbf{R},t) \cdot \mathbf{u}' = \frac{q}{\lambda(\mathbf{R},t)} \cdot \frac{1}{|\mathbf{d}'(\mathbf{R},t)|^2} - \frac{q \cdot v'^2 \cdot \cos^2(\theta)}{c^2 \cdot \lambda(\mathbf{R},t)} \cdot \frac{1}{|\mathbf{d}'(\mathbf{R},t)|^2} = \frac{q}{\lambda(\mathbf{R},t)} \cdot \frac{1}{|\mathbf{d}'(\mathbf{R},t)|^2} \cdot \left(1 - \frac{v'^2 \cdot \cos^2(\theta)}{c^2}\right) = \frac{q}{|\mathbf{d}'(\mathbf{R},t)|^2} \cdot \left(1 + \frac{v' \cdot \cos(\theta)}{c}\right) \qquad (35)$$

whereas the tangential field becomes

$$E_t = |\mathbf{E}(\mathbf{R},t) \times \mathbf{u}'| = \frac{q}{|\mathbf{d}'(\mathbf{R},t)|^2} \cdot \frac{v'^2}{c^2} \cdot \frac{|\cos(\theta) \cdot \sin(\theta)|}{\lambda(\mathbf{R},t)} \qquad (36)$$

This is in stark contrast to the fields obtained from the usual Liénard-Wiechert potentials (see Appendix A.3)

5. CONCLUSIONS AND OUTLOOK

The present investigation has shown that the well known Liénard-Wiechert potentials in the theory of electrodynamics are the result of failing to normalize the retarded particle distribution functions i.e. of violating the law of charge conservation in case of moving charges. With a properly normalized distribution function, the resultant expression for the scalar potential of a moving charge does not contain an explicitly velocity dependent factor but differs from that of a charge at rest only by the circumstance that the particle positions have to be taken at the retarded time rather than the instantaneous one.. And the resulting electric field does consequently not show the same divergent behaviour for charges approaching the speed of light as the corresponding solutions based on the Liénard-Wiechert potential. It should be emphasized that of course the validity of Maxwell's equations as such is not compromised by this, as in fact the results of this paper have been obtained on their basis (a recent paper by Heras [7] demonstrates indeed that a general discussion of the retarded potentials as formal solutions of Maxwell's equations can be made without specifying their exact form explicitly). .The present result should however be of strong relevance regarding specific interpretations of Maxwell's equations and indeed the theoretical interpretation of many problems in electrodynamics, in particular the relativistic kinematics of charged particles A detailed examination of such consequences is however beyond the scope of the present paper.

6. REFERENCES

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APPENDIX A.1

DERIVATION OF ELECTRIC FIELD OF MOVING CHARGE

a) FIELD DUE TO SCALAR POTENTIAL

The first contribution to the electric field comes from the gradient of the scalar potential (28)

$$\nabla_{\mathbf{R}} \phi(\mathbf{R},t) = \nabla_{\mathbf{R}} \frac{q}{|\mathbf{d'}(\mathbf{R},t)|} \tag{A.1.1}$$

With

$$d' = |\mathbf{d'}(\mathbf{R},t)| \tag{A.1.2.}$$

we have thus (using the chain rule)

$$\nabla_{\mathbf{R}} \phi(\mathbf{R},t) = \frac{\partial \phi}{\partial d'} \cdot \nabla_{\mathbf{R}} d' = \frac{-q}{d'^2} . \nabla_{\mathbf{R}} d' \tag{A.1.3}$$

With the definitions

$$\mathbf{R} = (X, Y, Z) \tag{A.1.4}$$

and

$$x' = X - s'_x, \quad y' = Y - s'_y, \quad z' = Z - s'_z \tag{A.1.5}$$

we have (see (30))

$$\mathbf{d'}(\mathbf{R},t) = (x', y', z') \tag{A.1.6}$$

and noting that

$$\nabla_{\mathbf{R}} = \left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \right) \tag{A.1.7}$$

and

$$d' = \sqrt{x'^2 + y'^2 + z'^2} \tag{A.1.8}$$

the x-component of $\nabla_{\mathbf{R}} d'$ can be written as (using the chain rule)

$$\frac{\partial d'}{\partial X} = \frac{1}{2d'} \cdot \left[2x' \left(1 - \frac{\partial s'_x}{\partial X} \right) - 2y' \frac{\partial s'_y}{\partial X} - 2z' \frac{\partial s'_z}{\partial X} \right] \tag{A.1.9}$$

Now because of the retardation condition

$$d' = c \cdot (t - t') \tag{A.1.10}$$

(where t' is the time emission and t the time of detection), x', y' and z' are only implicitly functions of the detector coordinate X, so we have to apply the chain rule

$$\frac{\partial s'_x}{\partial X} = \frac{\partial s'_x}{\partial t'} \cdot \frac{\partial t'}{\partial X} = \frac{-v'_x}{c} \frac{\partial d'}{\partial X} \tag{A.1.11}$$

$$\frac{\partial s'_y}{\partial X} = \frac{\partial s'_y}{\partial t'} \cdot \frac{\partial t'}{\partial X} = \frac{-v'_y}{c} \frac{\partial d'}{\partial X} \tag{A.1.12}$$

$$\frac{\partial s'_z}{\partial X} = \frac{\partial s'_z}{\partial t'} \cdot \frac{\partial t'}{\partial X} = \frac{-v'_z}{c} \frac{\partial d'}{\partial X} \tag{A.1.13}$$

where we have used the fact that $\partial s'_x / \partial t'$, $\partial s'_y / \partial t'$ and $\partial s'_z / \partial t'$ are nothing but the components of the (retarded) velocity of the charged particle.

After inserting (A.1.11)-(A.1.13) into (A.1.9) we obtain after a little algebra using the fact that (see (A.1.6))

$$x' \cdot v'_x + y' \cdot v'_y + z' \cdot v'_z = \mathbf{d'} \cdot \mathbf{v'} \tag{A.1.14}$$

$$\frac{\partial d'}{\partial X} = \frac{x'}{d'} + \frac{\mathbf{d'} \cdot \mathbf{v'}}{d' \cdot c} \cdot \frac{\partial d'}{\partial X} \tag{A.1.15}$$

or

$$\frac{\partial d'}{\partial X} = \frac{x'}{d'} \cdot \frac{1}{1 - \frac{\mathbf{u'} \cdot \mathbf{v'}}{c}} \tag{A.1.16}$$

with

$$\mathbf{u'} = \frac{\mathbf{d'}}{d'} \tag{A.1.17}$$

the unit vector from the retarded position of the charge to the observation point.

Analogously we get for the Y and Z components of the gradient of d'

$$\frac{\partial d'}{\partial Y} = \frac{y'}{d'} \cdot \frac{1}{1 - \frac{\mathbf{u}' \cdot \mathbf{v}'}{c}} \qquad \text{(A.1.18)}$$

$$\frac{\partial d'}{\partial Z} = \frac{z'}{d'} \cdot \frac{1}{1 - \frac{\mathbf{u}' \cdot \mathbf{v}'}{c}} \qquad \text{(A.1.19)}$$

and thus

$$\nabla_{\mathbf{R}} d' = \frac{\mathbf{u}'}{1 - \frac{\mathbf{u}' \cdot \mathbf{v}'}{c}} \qquad \text{(A.1.20)}$$

With this (A.1.3).becomes

$$\nabla_{\mathbf{R}} \phi(\mathbf{R},t) = \frac{-q}{1 - \frac{\mathbf{u}' \cdot \mathbf{v}'}{c}} \cdot \frac{\mathbf{d}'}{d'^3} \qquad \text{(A.1.21)}$$

b) FIELD DUE TO VECTOR POTENTIAL

According to Maxwell's equations, the second contribution to the electric field arises from the time derivative of the vector potential (see (25)). Since the vector and scalar potentials of single particle with (retarded) velocity $\mathbf{v}'$ are related by (see (28),(29))

$$\mathbf{A}(\mathbf{R},t) = \frac{\mathbf{v}'}{c} \cdot \phi(\mathbf{R},t) \qquad \text{(A.1.22)}$$

we have

$$\frac{1}{c} \cdot \frac{\partial \mathbf{A}(\mathbf{R},t)}{\partial t} = \frac{1}{c^2} \cdot \frac{\partial \left(\mathbf{v}' \cdot \phi(\mathbf{R},t) \right)}{\partial t} =$$

$$= \frac{1}{c^2} \cdot \left(\phi(\mathbf{R},t) \cdot \frac{\partial \mathbf{v}'}{\partial t} + \mathbf{v}' \cdot \frac{\partial \phi(\mathbf{R},t)}{\partial t} \right) =$$

$$= \frac{q}{c^2} \cdot \left(\frac{1}{d'} \cdot \frac{\partial \mathbf{v}'}{\partial t} - \mathbf{v}' \cdot \frac{1}{d'^2} \frac{\partial d'}{\partial t} \right) \qquad \text{(A.1.23)}$$

where the chain rule was applied to the second term in the bracket.
In the way of a total time derivative, the first term in the bracket describes the change of the vector potential due a change of the current (i.e. $\mathbf{v}'$) at constant distance (i.e. d') whereas the second term describes the change of the vector potential due to a change of the distance of the current system at constant current.
Therefore, the derivative $\partial \mathbf{v}' / \partial t$ in the bracket has to be taken at fixed d', which means the light travel time is fixed as well, so from (A.1.10) we can conclude that for this term any variation in the time of detection must equal the variation in the emission time i.e. $\partial t = \partial t'$ and thus

$$\frac{\partial \mathbf{v}'}{\partial t} = \frac{\partial \mathbf{v}'}{\partial t'} \qquad \text{(A.1.24)}$$

For the second term in the bracket in (A.1.23), the time derivative of d' is easily obtained from (A.1.8) using again the chain rule

$$\frac{\partial d'}{\partial t} = \frac{1}{2d'} \left(-2x' \cdot \frac{\partial s'_x}{\partial t} - 2y' \cdot \frac{\partial s'_y}{\partial t} - 2z' \cdot \frac{\partial s'_z}{\partial t} \right) =$$

$$= \frac{1}{2d'} \left(-2x' \cdot \frac{\partial s'_x}{\partial t'} - 2y' \cdot \frac{\partial s'_y}{\partial t'} - 2z' \cdot \frac{\partial s'_z}{\partial t'} \right) \cdot \frac{\partial t'}{\partial t} =$$

$$= -\mathbf{u}' \cdot \mathbf{v}' \cdot \frac{\partial t'}{\partial t} \qquad \text{(A.1.25)}$$

On the other hand, by differentiating the 'light equation' (A.1.10) with regard to t we obtain

$$\frac{\partial t'}{\partial t} = 1 - \frac{1}{c} \cdot \frac{\partial d'}{dt} \qquad \text{(A.1.26)}$$

so (A.1.25) becomes

$$\frac{\partial d'}{\partial t} = \frac{-\mathbf{u} \cdot \mathbf{v}}{1 - \frac{\mathbf{u}' \cdot \mathbf{v}'}{c}} \qquad \text{(A.1.27)}$$

Inserting (A.1.24) and (A.1.27) into (A.1.23) we have therefore (with $\mathbf{a}' = \partial \mathbf{v}' / \partial t'$)

$$\frac{1}{c} \cdot \frac{\partial \mathbf{A}(\mathbf{R},t)}{\partial t} = \frac{q}{c^2} \cdot \left(\frac{\mathbf{a}'}{d'} + \frac{1}{1 - \frac{\mathbf{u}' \cdot \mathbf{v}'}{c}} \cdot \frac{\mathbf{v}' \cdot (\mathbf{u}' \cdot \mathbf{v}')}{d'^2} \right) \qquad \text{(A.1.28)}$$

APPENDIX A.2

ELECTRIC FLUX FOR MOVING CHARGES
(GAUSS' LAW)

In order to check the consistency of the solution (34) for the electric field of a moving charge with Maxwell's equations, we first and foremost have to calculate whether the resultant flux through a closed surface complies with Gauss law

$$\frac{1}{4\pi}\oiint_{4\pi} dS \cdot \mathbf{n} \cdot \mathbf{E} = q \qquad \text{(A.2.1)}$$

where $\mathbf{n}$ is a unit vector normal to the surface element dS at that point.
Since generally the surface element dS is related to the solid angle element $d\Omega$ over the radial distance R by

$$dS = R^2 \cdot d\mathbf{\Omega} \qquad \text{(A.2.2)}$$

we can rewrite (A.2.1) as

$$\frac{1}{4\pi}\oiint_{4\pi} d\Omega \cdot R^2 \cdot \mathbf{n} \cdot \mathbf{E} = q \qquad \text{(A.2.3)}$$

We can simplify the integral significantly by choosing, without loss of generality, a spherical surface and assuming the charge producing the electric field at the origin. In this case the retarded distance $|\mathbf{d}'(R,t)|$ in the expression for the electric field (34) has the same constant value R anywhere on the sphere, and by choosing the polar axis along the directions of $\mathbf{d}'(R,t)$, $\mathbf{v}'$ and $\mathbf{a}'$ respectively for the 3 terms in (34) we can restrict ourselves to the integration over the polar angle θ due to the azimuthal symmetry, i.e.

$$\frac{R^2}{2}\int_0^{\pi} d\theta \cdot \sin(\theta) \cdot [F_{\mathbf{d}'}(\theta) + F_{\mathbf{v}'}(\theta) + F_{\mathbf{a}'}(\theta)] = q \qquad \text{(A.2.4)}$$

where

$$F_{\mathbf{d}'}(\theta) = \frac{q}{R^2} \cdot \frac{1}{1 - \frac{v'}{c}.\cos(\theta)} \qquad \text{(A.2.5)}$$

$$F_{\mathbf{v}'}(\theta) = -\frac{q}{R^2} \cdot \frac{v'^2}{c^2} \frac{\cos^2(\theta)}{1 - \frac{v'}{c}.\cos(\theta)} \qquad \text{(A.2.6)}$$

$$F_{\mathbf{a}'}(\theta) = -\frac{q}{R} \cdot \frac{a' \cdot \cos(\theta)}{c^2} \qquad \text{(A.2.7)}$$

Inserting these expressions into (A.2.4), it is immediately obvious that the acceleration term does not contribute anything to the total electric flux as the integral over $\sin(\theta) \cdot \cos(\theta)$ from 0 to π is zero. For the sum of the other two terms we can write

$$F_{\mathbf{d}'}(\theta) + F_{\mathbf{v}'}(\theta) = \frac{q}{R^2} \cdot \frac{1 - \frac{v'^2}{c^2} \cdot \cos^2(\theta)}{1 - \frac{v'}{c'} \cdot \cos(\theta)} =$$

$$= \frac{q}{R^2} \cdot \left(1 + \frac{v'}{c'} \cdot \cos(\theta)\right) \qquad \text{(A.2.8)}$$

However, the second term in the bracket integrates to zero as well in (A.2.4) for the same reason as above, so we are just left with the flux due to the static field and Gauss' law is thus fulfilled.(which it would not be if the velocity dependent factor $\partial t' / \partial t = 1/(1 - \mathbf{v}' \cdot \cos(\theta)/c)$ (see Appendix A1.b) had been multiplied to the acceleration term $F_{\mathbf{a}'}(\theta)$ term as well, as then the total flux for this term would not integrate to zero as required.).

APPENDIX A.3

ELECTRIC FIELD FOR UNNORMAIZED DISTRIBUTION FUNCTION

As explained in sects. 2 and 3, the usual derivation of the scalar an vector potentials (as found in the literature throughout, starting with Liénard [1] and Wiechert [2]) has an erroneous factor

$$\frac{1}{\lambda(\mathbf{R},t)} = \frac{1}{1-\frac{\mathbf{u}'\cdot\mathbf{v}}{c}} = \frac{1}{1-\frac{v'\cdot\cos(\theta)}{c}} \qquad \text{(A.3.1)}$$

that violates charge conservation as it makes the total charge aspect (i.e. position) dependent., so in this case the potentials are taken as

$$\phi_{nn}(\mathbf{R},t) = \frac{q}{\lambda(\mathbf{R},t)}\cdot\frac{1}{|\mathbf{d}'(\mathbf{R},t)|} \qquad \text{(A.3.2)}$$

$$\mathbf{A}_{nn}(\mathbf{R},t) = \frac{q}{\lambda(\mathbf{R},t)}\cdot\frac{\mathbf{v}'}{c}\cdot\frac{1}{|\mathbf{d}'(\mathbf{R},t)|}\cdot \qquad \text{(A.3.3)}$$

where the subscript 'nn' shall indicate that the density distribution is not normalized i.e. violates charge conservation.

We state here without going into the details of the calculation that in this case, following essentially the same procedure of calculating the spatial and time derivatives by applying the chain and product rules accordingly (only this time including the additional factor $1/\lambda(\mathbf{R},t)$), one obtains for the electric field

$$\mathbf{E}_{nn}(\mathbf{R},t) = \frac{q}{\lambda^3(\mathbf{R},t)}\cdot\frac{1}{|\mathbf{d}'(\mathbf{R},t)|^2}\cdot$$

$$\cdot\left[\left(\mathbf{u}'-\frac{\mathbf{v}'}{c}\right)\cdot\left(1-\frac{v'^2}{c^2}+\frac{\mathbf{d}'\cdot\mathbf{a}'}{c^2}\right)-\frac{d'\cdot\mathbf{a}'}{c^2}\cdot\left(1-\frac{\mathbf{u}'\cdot\mathbf{v}'}{c}\right)\right] \qquad \text{(A.3.4)}$$

This is the expression usually found in the literature i.e. also including the factor $1/\lambda(\mathbf{R},t)$ incorrectly applied to the acceleration terms when doing the derivatives of the product terms involving $\mathbf{v}'$ (as discussed in Appendix A.1.b.).

Even though (A.3.4) may be consistent with Gauss' law, this can at best be taken as an average statement regarding charge invariance, which however as such is only a necessary condition but by no means a sufficient one. And the erroneous factor $1/\lambda(\mathbf{R},t)$ in the usual Liénard-Wiechert potential simply invalidates the required independence of the total charge as a function of the aspect angle θ .

Apart from the incorrect additional factor $1/\lambda^2(\mathbf{R},t)$ applied to the overall amplitude of the electric field, (A.3.4) also obtains a spurious linear 'aberration' term $\frac{\mathbf{v}'}{c}$ added to the location unit vector $\mathbf{u}'$ even if the induction terms (which only cause higher order corrections) are ignored.. This is frequently interpreted in the literature to the effect that the retarded electric field points to the present positions of the particles rather than the retarded positions (see for instance Griffiths [4] or Jackson [3]), which is a logical contradiction in terms only caused by the erroneous derivation of the Liénard-Wiechert potentials (A.3.2) and (A.3.3) in the first place.

In contrast to the electric field obtained from the properly normalized charge distributions (see (35), (36)), the electric field based on the usual Liénard-Wiechert potentials shows a diverging behaviour as $\mathbf{v}'$ approaches c, as the radial field becomes (ignoring the acceleration terms, which vanish anyway)

$$E_{nn,r} = \mathbf{E}_{nn}(\mathbf{R},t)\cdot\mathbf{u}' = \frac{q}{\lambda^3(\mathbf{R},t)}\cdot\frac{1}{|\mathbf{d}'(\mathbf{R},t)|^2}\cdot$$

$$\cdot\left(1-\frac{v'\cdot\cos(\theta)}{c}\right)\cdot\left(1-\frac{v'^2}{c^2}\right) =$$

$$= \frac{q}{|\mathbf{d}'(\mathbf{R},t)|^2}\cdot\frac{1-\frac{v'^2}{c^2}}{\left(1-\frac{v'\cdot\cos(\theta)}{c}\right)^2} \qquad \text{(A.3.5)}$$

which obviously diverges for $\theta = 0$ as v' approaches c.(a direct result of the fact that the charge distribution is not normalized for the Liénard-Wiechert potential and incorrectly depends on the velocity through the term $1/\lambda(\mathbf{R},t)$. }

For the transverse electric field (A.3.4) yields instead (again neglecting the acceleration terms)

$$E_{nn,t} = |\mathbf{E}_{nn}(\mathbf{R},t)\times\mathbf{u}'| =$$

$$= \frac{q}{|\mathbf{d}'(\mathbf{R},t)|^2}\cdot\frac{|v'\cdot\sin(\theta)|}{c}\cdot\frac{1-\frac{v'^2}{c^2}}{\left(1-\frac{v'\cdot\cos(\theta)}{c}\right)^3} \qquad \text{(A.3.6)}$$

APPENDIX A.4

SCALE TRANSFORMATIONS OF PARTICLE DISTRIBUTION FUNCTIONS

The essential feature of the derivation of the Liénard-Wiechert potential is the apparent scale transformation of a particle distribution by a factor

$$\frac{1}{\lambda} = \frac{1}{1 - \frac{v'}{c}.\cos(\theta)} \qquad \text{(A.4.1)}$$

where θ is the angle between the velocity vector $\mathbf{v}'$ of the particle distribution (assumed to be uniform) and the line of sight to the observer.

Whilst the change of the overall extension of the particle distribution is recognized in these derivations, it is assumed (see for instance Sect. 10.3.1 in Griffiths [4]) that the particle density is somehow unaffected by this scale transformation, i.e. in the resulting integral for the scalar potential

$$\phi(\mathbf{R},t) = \frac{q}{\lambda} \cdot \int d^3\mathbf{r} \frac{\rho'(\mathbf{r},t)}{|\mathbf{R} - \mathbf{r}|} \qquad \text{(A.4.2)}$$

the retarded number density $\rho'(\mathbf{r},t)$ the observer at $\mathbf{R}$ sees at time t is assumed to be equal to the unretarded density measured locally at $\mathbf{r}$ at the retarded time t', i.e. it is assumed

$$\rho'(\mathbf{r},t) = \rho(\mathbf{r},t') \qquad \text{(A.4.3)}$$

and if the density is not explicitly time dependent

$$\rho'(\mathbf{r}) = \rho(\mathbf{r}) \quad . \qquad \text{(A.4.4)}$$

resulting consequently in an observer dependent total charge

$$\int d^3\mathbf{r}\, \rho'(\mathbf{r}) = \frac{N \cdot q}{\lambda} \qquad \text{(A.4.5)}$$

with N the total number of charges in the volume.
(note that in Griffiths [4] , ρ is defined as the charge density rather than the number density, so no factor N appears there).

It is of course clear what causes here this violation of charge conservation: the particle density $\rho(\mathbf{r})$ is in reality not a given continuous function but is defined in probabilistic way through the number of particles in a certain volume around point $\mathbf{r}$. And a scale transformation applied to the whole distribution of particles (and not only the end points), will obviously affect the distance between the particles and thus, by definition, also the density $\rho(\mathbf{r})$. For the one-dimensional case, a corresponding illustration was given already in sect. 3 of this paper

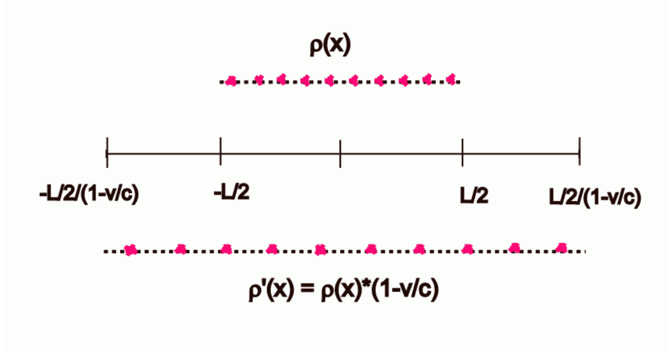

Fig.1 : Retarded vs. unretarded density distribution

The usual derivation of the Liénard-Wiechert potential on the other hand implies the following transformation for the density distribution $\rho(x)$

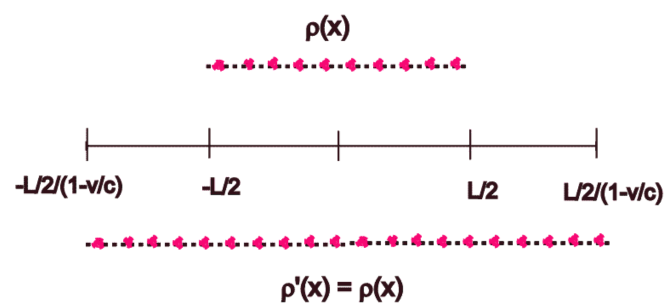

Fig.2 : Liénard-Wiechert retarded/unretarded density distributions

which, by assuming the scale transformation does not affect the density, has created additional charges out of nothing.

Whilst in some derivations, the assumption of the density being unaffected by the scale transformation is being made only implicitly (with the issue of the overall particle/charge conservation not being addressed at all), for instance Griffiths [4] and Aguirregabiria et al. [6] explicitly claim that the scale transformation due to the retardation effect would only affect the apparent volume but not the apparent density. Yet this claim is actually not a result of their derivation (as they seem to believe) but an unfounded (and indeed inconsistent) assumption.

It is thus clear that using a continuous density distribution function to approximate the actually discrete distribution of particles requires a normalization factor λ multiplied to the scale transformed density $\rho'(\mathbf{r},t)$ in order to preserve the total number of charges in (A.4.5). And with this, the factor $1/\lambda$ disappears from the equation (A.4.2) for the potential as well.

The same argument applies of course to any probability density function in general. The overall probability must stay normalized to 1 even after a scale transformation of the function. For instance, if we have the one-dimensional probability density function $p(x)$ satisfying (by definition) the normalization condition

$$\int dx\; p(x) = 1 \qquad \text{(A.4.6)}$$

and apply the scale factor λ to the argument of p, the resulting transformed probability density distribution $p'(\lambda \cdot x)$ must satisfy

$$\int dx\; p'(\lambda \cdot x) = 1 \qquad \text{(A.4.7)}$$

We can relate the transformed distribution to the original one by setting

$$p'(\lambda \cdot x) = \mu \cdot p(\lambda \cdot x) \qquad \text{(A.4.8)}$$

where μ is some (as yet unknown) normalization factor.
Inserting this into (A.4.7) yields

$$\mu \cdot \int dx\; p(\lambda \cdot x) = 1 \qquad \text{(A.4.9)}$$

and after substituting

$$x' = \lambda \cdot x \qquad \text{(A.4.10)}$$

$$\frac{\mu}{\lambda} \cdot \int dx'\; p(x') = 1 \qquad \text{(A.4.11)}$$

so because of (A.4.6) we have

$$\mu = \lambda \qquad \text{(A.4.12)}$$

i.e. from (A.4.8)

$$p'(\lambda \cdot x) = \lambda \cdot p(\lambda \cdot x) \qquad \text{(A.4.13)}$$

The appearance of the scale factor as a normalization constant is of course a well-known property for probability distribution functions in general, and it is only its neglection that leads to the Liénard-Wiechert potential. Correctly, the expression for the potential should have been obtained by integrating over $\rho'(\mathbf{r},t) = \lambda \cdot \rho(\mathbf{r},t')$ rather than $\rho(\mathbf{r},t')$ so instead of (A.4.2) the correct equation for the potential is in fact

$$\phi(\mathbf{R},t) = \frac{q}{\lambda} \cdot \int d^3\mathbf{r}\, \frac{\lambda \cdot \rho'(\mathbf{r},t)}{|\mathbf{R} - \mathbf{r}|} \qquad \text{(A.4.14)}$$

which means the factor λ obviously cancels out. (as the density change due to the scale transformation exactly cancels out the volume change for the particle distribution)..

And the above argument can be generalized in a straightforward manner to non-constant scale factors as well as multivariate functions on the basis of the formal equations developed in sect.2,